UNIVERSITÄT STUTTGART
LEHRSTUHL FÜR SYSTEMTHEORIE UND SIGNALVERARBEITUNG
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Written exam in
Digital Signal Processing 09/20/2010
8.00–9.30 h 3 Problems

Solutions to Problem 1: Time Domain and z-Transform (30 Points)

1.1 The filter is time invariant, because T doesn't depend on time n.
The filter is causal, because the output $y(n)$ depends only on present and former values of $x(n)$.
The filter is unstable because (it's causal and) at least one pole of the characteristic polynomial is outside of the unit circle:

$$\lambda^2 - 4\lambda + 3 = 0 \Leftrightarrow \lambda \in \{3, 1\}$$

3

1.2 It is:

$$\begin{aligned} y_k = T(x_k(n)) &: y_k(n) - 4y_k(n-1) + 3y_k(n-2) = 6x_k(n) - 6x_k(n-1) \\ x(n) = x_1(n) &: y_1(n) - 4y_1(n-1) + 3y_1(n-2) = 6x_1(n) - 6x_1(n-1) \\ x(n) = x_2(n) &: y_2(n) - 4y_2(n-1) + 3y_2(n-2) = 6x_2(n) - 6x_2(n-1) \end{aligned}$$

Defining

$$\begin{aligned} &\tilde{x}(n) := x_1(n) + x_2(n) \\ \Rightarrow\ &\tilde{y}(n) = T(\tilde{x}(n)) \end{aligned}$$

and using the LHS from above:

$$\begin{aligned} \Leftrightarrow\ \tilde{y}(n) - 4\tilde{y}(n-1) + 3\tilde{y}(n-2) &= 6(x_1(n) + x_2(n)) - 6(x_1(n-1) + x_2(n-1)) \\ ... &= \underbrace{6(x_1(n) - x_1(n-1))}_{y_1(n)-4y_1(n-1)+3y_1(n-2)} + \underbrace{6(x_2(n) - x_2(n-1))}_{y_2(n)-4y_2(n-1)+3y_2(n-2)} \\ \tilde{y}(n) - 4\tilde{y}(n-1) + 3\tilde{y}(n-2) &= (y_1 + y_2)|_n - 4(y_1 + y_2)|_{n-1} + 3(y_1 + y_2)|_{n-2} \\ \Rightarrow\ \tilde{y}(n) &= y_1(n) + y_2(n) \text{ qed} \end{aligned}$$

4

1.3 Solving the difference equation:

- Homogeneous part: Inserting the ansatz $y_h = z^n$ leads to

  $$z^n(1 - 4z^{-1} + 3z^{-2}) = 0 \text{ and with } z^n \neq 0 \ \Rightarrow z_1 = 3,\ z_2 = 1$$

  Solution: $y_h(n) = c_1(3)^n + c_2(1)^n$.
  Alternative: char. polynomial and simple insert in general solution formula.

- Particular part:
  An ansatz according to RHS: $y_p = c_3 \cdot \left(\frac{1}{3}\right)^n$ is used.
  LHS: With ansatz for all n

$$y_p(n) - 4y_p(n-1) + 3y_p(n-2) = c_3 \cdot \left(\frac{1}{3}\right)^n \cdot \left(1 - 4\left(\frac{1}{3}\right)^{-1} + 3\left(\frac{1}{3}\right)^{-2}\right) = 16c_3 \cdot \left(\frac{1}{3}\right)^n$$

  RHS: By inserting $x(n) = \left(\frac{1}{3}\right)^n$, $n \geq 0$, it is

$$6 \cdot \left(\frac{1}{3}\right)^n - 6x(n-1)$$

  The equation with LHS and RHS has to hold $\forall n \geq 0$. Picking $n = 0$ for evaluation and using given IC $x(-1) = 3$ gives

$$\begin{aligned} 16c_3 \cdot \left(\frac{1}{3}\right)^0 &= 6 \cdot \left(\frac{1}{3}\right)^0 - 6x(-1) \\ 16c_3 &= 6 - 18 \end{aligned}$$

  and leads to $c_3 = -\frac{3}{4}$.

- Joint parametric solution for $n \geq 0$:

$$y(n) = c_1(3)^n + c_2(1)^n - \frac{3}{4}\left(\frac{1}{3}\right)^n$$

- Parameters from initial conditions:

$$\begin{aligned} y(-1) = 1 &\overset{!}{\Leftrightarrow} \frac{c_1}{3} + c_2 \cdot 1 - \frac{9}{4} \\ y(-2) = 0 &\overset{!}{\Leftrightarrow} \frac{c_1}{9} + c_2 \cdot 1 - \frac{27}{4} \end{aligned}$$

  solves to

$$c_1 = -\frac{63}{4},\ c_2 = \frac{17}{2}.$$

- Solution:

$$y(n) = -\frac{63}{4}(3)^n - \frac{3}{4}\left(\frac{1}{3}\right)^n + \frac{17}{2}$$

10

1.4 z-Transform: w. vanishing initial conditions for general input x

$$\begin{aligned} &T : y(n) - 4y(n-1) + 3y(n-2) = 6x(n) - 6x(n-1) \\ &\overset{\mathcal{Z}}{\rightarrow} Y(z) - 4z^{-1}Y(z) + 3z^{-2}Y(z) = 6X(z) - 6z^{-1}X(z) \\ &T(z) = \frac{Y(z)}{X(z)} = \frac{6 - 6z^{-1}}{1 - 4z^{-1} + 3z^{-2}} \end{aligned}$$

2

1.5 Nominator and denominator polynomials in the transfer function

$$H(z) = \frac{z^3 + -\frac{5}{2}z^2 + 3z - 1}{z^4 + \frac{5}{3}z^3 + \frac{7}{6}z^2 + \frac{1}{3}z} = \frac{z^{-1} + -\frac{5}{2}z^{-2} + 3z^{-3} - z^{-4}}{1 + \frac{5}{3}z^{-1} + \frac{7}{6}z^{-2} + \frac{1}{3}z^{-3}}$$

3

1.6 The filter is stable, because all poles are within the unit cycle. 2

1.7 A system has minimum phase, if all poles and zeros are within the unit circle. 2

1.8 The decomposition is:

$$H(z) = H_{\min}(z) \cdot H_{\text{all}}(z)$$

$$H_{\min}(z) = \frac{(1 - \frac{1}{2}z^{-1})z^{-1} \cdot (1 - \frac{1}{\sqrt{2}}e^{j\cdot\frac{\pi}{4}} \cdot z^{-1})(1 - \frac{1}{\sqrt{2}}e^{-j\cdot\frac{\pi}{4}} \cdot z^{-1})}{\prod_i (1 - p_i \cdot z^{-1})}$$

$$H_{\text{all}}(z) = \frac{(1 - \sqrt{2}e^{j\cdot\frac{\pi}{4}} \cdot z^{-1})(1 - \sqrt{2}e^{j\cdot-\frac{\pi}{4}} \cdot z^{-1})}{(1 - \frac{1}{\sqrt{2}}e^{j\cdot\frac{\pi}{4}} \cdot z^{-1})(1 - \frac{1}{\sqrt{2}}e^{-j\cdot\frac{\pi}{4}} \cdot z^{-1})}$$

4

problem	points
1.1	3
1.2	4
1.3	10
1.4	2
1.5	3
1.6	2
1.7	2
1.8	4
sum	30

Solutions to Problem 2: DTFT (30 Points)

2.1 In general $H(e^{j\omega_r})$ is complex, as $h(n)$ is not symmetrical (around $n = 0$). 2

2.2 Investigating spatial effects, looking at the causality does not make sense, as systems with negative n (positions) are implementable. Causality is a temporal property. 2

2.3 a)

$$H\left(e^{j\omega_r}\right) = 3e^{-j\omega_r} + 2e^{-j2\omega_r} + 4e^{-j3\omega_r} + 3e^{-j4\omega_r}$$

2

b)

$$H\left(e^{j0}\right) = H(1) = 12$$

1

c)

$$H\left(e^{j\frac{\pi}{2}}\right) = 3e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + 4e^{-j\frac{3\pi}{2}} + 3e^{-j2\pi} = -3j - 2 + 4j + 3 = 1 + j$$

2

2.4 a)

$$H_F(e^{j\omega_r}) = 3e^{-4j\omega_r} + 2e^{-j3\omega_r} + 4e^{-j2\omega_r} + 3e^{-j\omega_r}$$

1

b) To get h from h_F, the sensor must be shifted, so that it is symmetrical to 0. Afterwards it can be mirrored and back-shifted. Alternatively it can be shifted to the left and mirrored afterwards.

$$h_F(n) = \left(h(n)\Big|_{n\to n-(-5)}\right)\Big|_{n\to -n} = h(-n+5)$$

$$\Rightarrow H_F(e^{j\omega_r}) = \left(e^{j\omega_r 5} H(e^{j\omega_r})\right)\Big|_{\omega_r\to-\omega_r} = e^{-j5\omega_r} H(e^{-j\omega_r})$$

3

c) Neither mirroring nor shifting change the absolute value, which can be calculated with the results of problem 1.3 easily.

$$\left|H(e^{j\frac{\pi}{2}})\right| = \left|H_F(e^{j\frac{\pi}{2}})\right| = |1 + j| = \sqrt{2}$$

2

2.5 a) The maximum is at $\omega_r = 0$. There, the complex arrows add constructively (coherently). As $|H(e^{j\omega_r})|$ is symmetrical, $\omega_r = 0$ must be a extremum. This reasoning is not enough, as it could be a minimum. 2

b) As $h(n)$ is real, $|H(e^{j\omega_r})|$ must be symmetrical around $\omega_r = 0$. 2

2.6 The integral can be calculated by the method of moments

$$\int_0^{2\pi} H(e^{j\omega_r})d\omega_r = h(0) = 0.$$

3

2.7

$$x(n) = \int_{-\pi}^{\pi} X\left(e^{j\omega_r}\right) e^{j\omega_r n} \frac{d\omega_r}{2\pi} = \frac{1}{2\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{j\omega_r n} d\omega_r = \frac{1}{2\pi j n} \left[e^{j\omega_r n}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2\pi j n}\left(e^{j\frac{1}{2}n} - e^{-j\frac{1}{2}n}\right) = \frac{1}{2\pi j n} 2j \sin\left(\frac{1}{2}n\right) = \frac{1}{\pi n} \sin\frac{n}{2}$$

4

$$x(1) = \frac{1}{\pi}\sin\frac{1}{2}; \quad x(2) = \frac{1}{2\pi}\sin 1; \quad x(3) = \frac{1}{3\pi}\sin\frac{3}{2}; \quad x(4) = \frac{1}{4\pi}\sin 2$$

2.8

$$y = x * h \Rightarrow Y\left(e^{j\omega_r}\right) = X\left(e^{j\omega_r}\right) H\left(e^{j\omega_r}\right)$$

$$Y\left(e^{j\omega_r}\right) = u\left(\omega_r + \frac{1}{2}\right) u\left(-\omega_r + \frac{1}{2}\right)\left(3e^{-j\omega_r} + 2e^{-j2\omega_r} + 4e^{-j3\omega_r} + 3e^{-j4\omega_r}\right)$$

4

problem	points
2.1	2
2.2	2
2.3	5
2.4	6
2.5	4
2.6	3
2.7	4
2.8	4
sum	30

Solutions to Problem 3: Fourier Transforms (30 Points)

Part I:

3.1

$$y = x(n) * h(n) = x(n)h(0) + x(n-1)h(1) + x(n-2)h(2) + x(n-3)h(3)$$
$$= \{\underset{\uparrow}{0}, 4, 12, 3, 3, 12, -9, 2\}$$
$$y_5 = x(n) \circledast h(n) = \{\underset{\uparrow}{12}, -5, 14, 3, 3\}$$

Same result als linear convolution: use zero-padding to length 7 [6]

3.2

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = 2 + 7e^{-j\omega} + 3e^{-j2\omega} - 4e^{-j3\omega} + e^{-j4\omega}$$
$$H\left(e^{j\omega}\right) = 2e^{-j\omega} - e^{-j2\omega} + 2e^{-j3\omega}$$
$$Y\left(e^{j\omega}\right) = 4e^{-j\omega} + 12e^{-j2\omega} + 3e^{-j3\omega} + 3e^{-j4\omega} + 12e^{-j5\omega} - 9e^{-j6\omega} + 2e^{-j7\omega}$$

[4]

3.3

$$H_4(k) = \sum_{n=0}^{3} h(n)e^{-j\frac{2\pi}{4}kn} = 2e^{-j\frac{2\pi}{4}k} - e^{-j\frac{4\pi}{4}k} + 2e^{-j\frac{6\pi}{4}k}$$
$$H_5(k) = \sum_{n=0}^{4} h(n)e^{-j\frac{2\pi}{5}kn} = 2e^{-j\frac{2\pi}{5}k} - e^{-j\frac{4\pi}{5}k} + 2e^{-j\frac{6\pi}{5}k}$$

[3]

3.4 The system described by $h(n)$ is a FIR filter, because the length of $h(n)$ is finite.
The system is stable, because it is an FIR system. [2]

3.5 $h(2) = 2$ and $h(3) = 0$ [2]

Part II:

3.6 $N = 64 \Rightarrow b = 6$

$$R_{\text{first}}(64) = \tilde{R}_{\text{first}}(6) = \frac{1}{27}\left((105 \cdot 6 - 123)2^6 - 54 \cdot 6 + 114\right)$$
$$= \frac{1}{27}((630 - 123)\,64 - 324 + 114) = \frac{1}{27}(507 \cdot 64 - 210)$$
$$= \frac{1}{27}(507 \cdot 63 + 507 - 210) = \frac{1}{27}(507 \cdot 63 + 297)$$
$$= \frac{507}{3} \cdot \frac{63}{9} + \frac{297}{27} = 169 \cdot 7 + 11 = 1183 + 11 = 1194$$

[2]

$$R_{\text{second}}(64) = 4 \cdot 64 \cdot 6 - 5 \cdot 64 - 22 = (24 - 5) \cdot 64 - 22 = 19 \cdot 64 - 22$$
$$= 20 \cdot 64 - 64 - 22 = 1280 - 86 = 1194$$

$\Rightarrow$ Both algorithms need the same number of operations for a 64-point FFT. 2

Asymptotic case: only largest summand relevant 1

$$\Rightarrow \lim_{N\to\infty} \frac{R_{\text{first}}(N)}{R_{\text{second}}(N)} = \frac{\frac{35}{9} N \log_2(N)}{4N \log_2(N)} = \frac{\frac{35}{9}}{4} = \frac{35}{36}$$

$\Rightarrow$ first algorithm is $\frac{1}{36} \approx 3\%$ faster for large N. 2

3.7 Thinking about the convolution, it becomes clear, that every sample has to be multiplied with every filter tap.

$$R_{\text{conv,complexmul}} = NM$$

The results have to be added, but only $M-1$ additions are needed to add M numbers. This needs to be carried out $N-1$ times, due to the fading areas at the beginning and the end (playing with simple examples reveals this).

$$R_{\text{conv,complexadd}} = (N-1)(M-1) = NM - M - N + 1$$

The number of real operations follows directly from these considerations

$$R_{\text{conv,realop}} = 6R_{\text{conv,complexmul}} + 2R_{\text{conv,complexadd}} = 6NM + 2NM - 2M - 2N + 2 = 8NM - 2M - 2N + 2$$

3

3.8 The fast convolution consists of zero padding, forward transform of the signal, multiplication with the transfer function and backward transform of the result.
The length of the signal must be increased to $N + M - 1$ by zero-padding to avoid aliasing, which needs no operations according to the assumptions.
The length of the FFT is the length of the zero-padded signal.

$$R_{\text{fastconv,realopFFT}} = R_{\text{second}}(N + M - 1) = 4(N + M - 1)\log_2(N + M - 1) - 5(N + M - 1) - 22$$

The transfer function $H(k)$ with length $N + M - 1$ is, according to the assumptions, already available. It must be multiplicated element-wise by the transformed signal, so $N + M - 1$ complex multiplications are needed.

$$R_{\text{fastconv,realopFreq}} = 6(N + M - 1)$$

As the length of the signal does not change in frequency domain, forward and backward transforms need the same number of operations. So the overall operations can be calculated.

$$\begin{aligned} R_{\text{fastconv,realop}} &= 2R_{\text{fastconv,realopFFT}} + R_{\text{fastconv,realopFreq}} \\ &= 8(N + M - 1)\log_2(N + M - 1) - 10(N + M - 1) - 44 + 6(N + M - 1) \\ &= 8(N + M - 1)\log_2(N + M - 1) - 4(N + M) - 40 \end{aligned}$$

3

problem	points
3.1	6
3.2	4
3.3	3
3.4	2
3.5	2
3.6	7
3.7	3
3.8	3
sum	30