UNIVERSITÄT STUTTGART
LEHRSTUHL FÜR SYSTEMTHEORIE UND SIGNALVERARBEITUNG
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Written exam in
Digital Signal Processing 09/20/2010
8.00–9.30 h 3 Problems

All resources are permitted except electronic calculators.

- Please write only on one side of each page.
- Please submit the cover sheet/envelope, not the problem sheets.

Problem 1: Time Domain and z-Transform (30 Points)

Consider the filter $y(n) = T(x(n))$ with the difference equation

$$\frac{1}{12}y(n) - \frac{1}{3}y(n-1) + \frac{1}{4}y(n-2) = \frac{1}{2}\left(x(n) - x(n-1)\right)$$

1.1 State briefly, if the filter is time-invariant, causal or stable.

1.2 Show that the superposition principle holds for the two general inputs $x_1(n)$ and $x_2(n)$.
Hint: $T\left(\sum_i c_i x_i(n)\right) = \sum_i c_i T\left(x_i(n)\right)$

1.3 Now, the input signal to this filter is given by

$$x(n) = \left(\frac{1}{3}\right)^n, \ n \geq 0$$

and initial conditions are defined as $y(-1) = 1,\ y(-2) = 0,\ x(-1) = 3$. Calculate the filter output $y(n)$.

1.4 Give the transfer function of the difference equation.

Hint: This part can be solved independently of the former.

Now, another filter H is to be examined. Its zeros z_k and poles p_i are given by

$$z_k = \left\{\frac{1}{2},\ 1 + j\cdot 1,\ 1 - j\cdot 1\right\},\ p_i = \left\{0,\ -\frac{2}{3},\ -\frac{1}{2} + j\cdot\frac{1}{2},\ -\frac{1}{2} - j\cdot\frac{1}{2}\right\} \tag{1}$$

1.5 Give the nominator and denominator polynomials of the transfer function $H(z)$ of (1).

1.6 Is this filter stable? State briefly.

The filter $H(z)$ is to be decomposed in a minimum-phase filter $H_{\text{min}}(z)$ and an allpass filter $H_{\text{all}}(z)$.

1.7 State briefly, what the conditions for the minimum-phase property are and why $H(z)$ doesn't fullfill them.

1.8 Give a minimum-phase filter $H_{\text{min}}(z)$ and an allpass filter $H_{\text{all}}(z)$ as a decomposition of $H(z)$.

Problem 2: DTFT (30 Points)

A radar sensor mounted on a car is investigated under far field conditions. The sensor has four differently sized patch antennas at equally spaced positions $n = \{1, 2, 3, 4\}$. The size of the antennas' elements specify their weighting. The weighting is

$$h(1) = 3, \quad h(2) = 2, \quad h(3) = 4, \quad h(4) = 3 \quad \text{and 0 otherwise.}$$

$h(n)$ can be interpreted as the radar sensor's spatial impulse response. The sensor's spatial transfer function of a point object is

$$H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}}) = \sum_{n=-\infty}^{\infty} h(n)\mathrm{e}^{-\mathrm{j}\omega_\mathrm{r} n}.$$

ω_r is the spatial frequency (comparable to temporal frequency), which can be used to determine the direction of the object.

2.1 Do you expect that $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ is a real or a complex (non-real) function for any $\omega_\mathrm{r} \in \mathbb{R}$? State *briefly*.

2.2 Does it make sense to demand that the radar sensor has a causal spatial impulse response? State *briefly* why or why not.

2.3
a) Calculate the spatial frequency response $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ explicitly.
b) Calculate $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ for a frontal object with the spatial frequency of $\omega_\mathrm{r} = 0$.
c) Calculate $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ for the spatial frequency of $\omega_\mathrm{r} = \frac{\pi}{2}$.

2.4 Due to a faulty assembling the sensor is mounted upside down. The antenna's elements are now

$$h_\mathrm{F}(1) = 3, \quad h_\mathrm{F}(2) = 4, \quad h_\mathrm{F}(3) = 2, \quad h_\mathrm{F}(4) = 3 \quad \text{and 0 otherwise.}$$

Let the transfer function of the faulty assembled radar sensor be $H_\mathrm{F}(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$.

a) Calculate $H_\mathrm{F}(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$.
b) What is the relationship between $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ and $H_\mathrm{F}(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$?
c) Calculate $\left|H_\mathrm{F}(\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})\right|$ and $\left|H(\mathrm{e}^{\mathrm{j}\frac{\pi}{2}})\right|$.

2.5 The absolute value of $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ determines the amplifying of the different directions.

a) Using the correctly mounted radar sensor, for which ω_r from $-\frac{\pi}{2} \leq \omega_\mathrm{r} \leq \frac{\pi}{2}$ has $H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})$ the largest amplification? State *briefly*.
b) Is there a symmetry in $|H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})|$? State *briefly*.

2.6 Calculate $\int_0^{2\pi} H(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}})\mathrm{d}\omega_\mathrm{r}$.

2.7 A specially formed spread object, which cannot be treated as a point object, is detected. The object can be described in the spatial frequency domain as

$$X\left(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}}\right) = u\left(\omega_\mathrm{r} + \frac{1}{2}\right)u\left(-\omega_\mathrm{r} + \frac{1}{2}\right),$$

where u is the unit step function. Calculate the discrete signal $x(n)$ in general and at the values $n = \{1, 2, 3, 4\}$.

2.8 Let $y = y(n)$ be the result of the convolution $y(n) = (x * h)(n)$. Calculate its spectrum $Y\left(\mathrm{e}^{\mathrm{j}\omega_\mathrm{r}}\right)$.

Problem 3: Fourier Transforms (30 Points)

Part I:

A signal $x(n)$ and a system with the impulse response $h(n)$ are given by

$$x(n) = \{\underset{\uparrow}{2}, 7, 3, -4, 1\}, \qquad h(n) = \{\underset{\uparrow}{0}, 2, -1, 2\}$$

3.1 Calculate $y = x(n) * h(n)$ and the 5-point circular convolution $y_5 = x(n) \circledast h(n)$.
Assume you want to do a 7-point circular convolution. What do you have to do before calculating the convolution to get the same result as with linear convolution?

3.2 Calculate the Fourier transforms $X\left(e^{j\omega}\right)$ of $x(n)$ and $H\left(e^{j\omega}\right)$ of $h(n)$ and their product $Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right)$

3.3 Calculate and simplify the 4-point DFT $H_4(k)$ and the 5-point DFT $H_5(k)$ of $h(n)$.

3.4 Is the system described by $h(n)$ a FIR or an IIR filter? Is it stable? State briefly.

3.5 Suppose you could change the values $h(2)$ and $h(3)$. Give the corresponding values so that $h(n)$ has a linear phase.

Part II:

A first *complex-valued* FFT/IFFT algorithm uses the following number of operations $R_{\text{first}}(N)$ depending on the FFT length $N = 2^b$, where every *real-valued* addition or multiplication is counted as one operation and b is an even integer number.

$$R_{\text{first}}(N) = \tilde{R}_{\text{first}}(b) = \frac{1}{27}\left((105b - 123)2^b - 54b + 114\right)$$

A second FFT algorithm needs the following number of operations for the same task:

$$R_{\text{second}}(N) = 4N\log_2(N) - 5N - 22$$

3.6 Compare both algorithms when calculating a 64-point FFT. Give the ratio of the number of operations in the asymptotic case (large N).

The convolution and the fast convolution by FFT of the complex signal $x(n)$ of length N with the complex impulse response $h(n)$ of length M should be compared. Therefor assume the following for the rest of this problem:

- every complex multiplication needs 6 real operations
- every complex addition needs 2 real operations
- zero-padding does not count as an operation
- no overlap-add or overlap-save are used
- all zero-padded forms of $h(n)$ and $H(k)$ are already available
- the *second* FFT/IFFT algorithm is used

3.7 Give the number of complex additions, complex multiplications, and overall real operations of the *conventional* convolution.

3.8 Give the number of real operations needed for the FFT, the number needed doing the calculations in the frequency domain, and the overall real operations of the *fast* convolution.