Direction of Arrival Estimation of Two Moving Targets Using a Time Division Multiplexed Colocated MIMO Radar

Kilian Rambach
Institute of Signal Processing and System Theory
University of Stuttgart, Germany
Email: kilian.rambach@iss.uni-stuttgart.de

Bin Yang
Institute of Signal Processing and System Theory
University of Stuttgart, Germany
Email: bin.yang@iss.uni-stuttgart.de

***Abstract*— A Multiple-Input-Multiple-Output (MIMO) radar with colocated transmit and receive antennas has a larger virtual aperture compared to the corresponding Single-Input-Single-Output (SIMO) radar. Therefore, it can achieve a more accurate Direction of Arrival (DOA) estimation. Due to the Doppler effect, a target moving relative to the radar system results in an additional phase shift of the baseband signal. In general, this leads to a decrease in the DOA estimation accuracy. We consider time division multiplexed (TDM) MIMO radars and derive the Cramer-Rao Bound (CRB) for the DOA and Doppler frequency estimation of two moving targets. This is done for general TDM schemes. This enables to compare the achievable accuracy for different TDM MIMO radars. We derive conditions for TDM schemes which lead to a decoupling of the Doppler frequencies and DOAs in the CRB. Hence a CRB of DOAs can be achieved which is as small as if the Doppler frequencies are known a priori. We define a statistical resolution limit to separate both targets with the help of the CRB and compare the resolution of a TDM MIMO radar to that of a SIMO radar.**

I. INTRODUCTION

MIMO radars have attained increased attention recently due to their advantages compared to SIMO radars, e.g. a higher number of detectable targets and higher accuracy in DOA estimation [1], [2]. Good performance in DOA estimation is important in many applications. In order to investigate the DOA performance of a radar system independently of the used algorithm, the CRB can be used. It is a lower bound on the covariance matrix of all unbiased estimators. The CRB for DOA estimation of stationary targets using a MIMO radar has been computed in [1], [3]. In order to distinguish the transmitted signal on the receiver side, different multiplexing techniques like code, frequency or time division multiplexing (TDM) can be used. In this paper, we focus on TDM MIMO radars. To estimate the DOA, the phases of the received baseband signals are analyzed. A target which moves relative to the radar system causes an additional phase shift in the baseband signal due to the Doppler effect. In general, this decreases the accuracy of the DOA estimation, since this Doppler phase shift has to be estimated as well. In [4] the CRB for the DOA estimation of one moving target for a TDM MIMO radar was computed. It was shown that the achievable accuracy depends on the TDM scheme used, i.e. on the sequence and the positions of the transmitting antennas. Optimal TDM schemes were derived which yield an accuracy which is as good as for a stationary target. Besides a good DOA estimation for one target, the estimation of the DOAs from the superimposed signals of two targets is an important application. This happens e.g. in automotive applications in which the targets can't be distinguished in their distance or velocity. It is unlikely that more than two targets have nearly the same distance and velocity, hence we focus on the two target case. Since a small difference in the Doppler frequencies causes already different phase shifts in the baseband signal, both Doppler frequencies as well as both DOAs have to be estimated in parallel. We investigate the performance of the DOA estimation of a MIMO radar for two moving targets. We compute for the first time the CRB for the DOA and Doppler frequency estimation for a TDM MIMO radar. We derive conditions for TDM schemes such that the DOAs and Doppler frequencies decouple in the CRB. Hence the DOA estimation is as accurate as if the Doppler frequencies are known a priori. Using the CRB, we define a statistical resolution limit to separate both targets. We show that the TDM MIMO radar can achieve a better resolution than the SIMO radar, despite the unknown Doppler frequencies.

We present the signal model in section II and derive the CRB in section III. In section IV, conditions for the decoupling of the DOAs and Doppler frequencies are presented. The theoretical computations are confirmed by simulations in section V. In section VI, the angular resolution limit is investigated.

We use the following notations in the paper: $\underline{1}_K$ is a vector of length K with all elements equal 1, and $\mathbf{I}$ is the identity matrix. * stands for conjugate, ^{T} for transpose and ^{H} for conjugate transpose. x_i denotes the i-th element of vector $\underline{x}$. $\underline{y} = \exp(\underline{x})$ and $\underline{z} = \sqrt{\underline{x}}$ are understood as element-by-element operations, i.e. $y_i = \exp(x_i)$ and $z_i = \sqrt{x_i}$, respectively. $\otimes$ is the Kronecker tensor product and $\odot$ the entrywise Hadamard product. $\operatorname{diag}(\underline{x})$ is the diagonal matrix consisting of the elements of $\underline{x}$.

II. Signal Model

We investigate a MIMO radar using time division multiplexing. It consists of a linear array with N_{Tx} transmitting (Tx) and N_{Rx} receiving (Rx) colocated, isotropic antennas. The transmitted signal is narrowband. The moving targets are modeled as point sources reflecting the transmitted signal. The targets are in the far field. The DOA of the targets, measured perpendicular to the linear array, is denoted by $\underline{\vartheta} = [\vartheta_1, \vartheta_2]^T$. $\underline{d}^{\text{Rx}} \in \mathbb{R}^{N_{\text{Rx}}}$ and $\underline{d}^{\text{Tx}} \in \mathbb{R}^{N_{\text{Tx}}}$ are the positions of the Rx and Tx antennas in units of $\frac{\lambda}{2\pi}$, respectively, where λ is the carrier wavelength. The Tx antennas transmit N_{Pulse} pulses at the time instances given by $\underline{t} \in \mathbb{R}^{N_{\text{Pulse}}}$. The energies per pulse are given by $\underline{\rho} \in \mathbb{R}^{N_{\text{Pulse}}}$, i.e. each element is the product of the transmitting power and the pulse duration. The positions of the Tx antennas in the sequence in which they transmit are denoted by $\underline{d}^{\text{Pulse}} \in \mathbb{R}^{N_{\text{Pulse}}}$. Note that some positions in $\underline{d}^{\text{Pulse}}$ can occur several times, if an antenna transmits more than once, see Fig. 1 for an example.

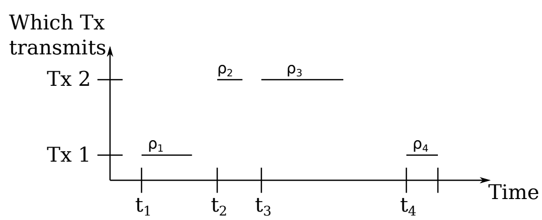

Fig. 1. Example of a TDM scheme: 2 transmitters transmitting at times $\underline{t} = [t_1, t_2, t_3, t_4]^T$ with energy $\underline{\rho} = [\rho_1, \rho_2, \rho_3, \rho_4]^T$. Here, $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, d_2^{\text{Tx}}, d_2^{\text{Tx}}, d_1^{\text{Tx}}]^T$.

We introduce first the signal model for one target with DOA ϑ and extend it to two targets afterwards. The steering vector of the Rx array is

$$\underline{a}^{\text{Rx}}(u) = \exp(j \cdot \underline{d}^{\text{Rx}} u) \tag{1}$$

with the electrical angle $u = \sin(\vartheta)$. The steering vector of the Tx array with the transmitting sequence $\underline{d}^{\text{Pulse}}$ is given by

$$\underline{a}^{\text{Pulse}}(u) = \exp(j \cdot \underline{d}^{\text{Pulse}} u). \tag{2}$$

The steering vector of the virtual array which includes all Rx-Tx combinations can be written as

$$\underline{a}^{\text{virt}}(u) = \underline{a}^{\text{Pulse}}(u) \otimes \underline{a}^{\text{Rx}}(u) = \exp(j \cdot \underline{d}^{\text{virt}} u) \tag{3}$$

with

$$\underline{d}^{\text{virt}} = \underline{1}_{N_{\text{Pulse}}} \otimes \underline{d}^{\text{Rx}} + \underline{d}^{\text{Pulse}} \otimes \underline{1}_{N_{\text{Rx}}} \in \mathbb{R}^{N_{\text{virt}}}, \tag{4}$$

$$N_{\text{virt}} = N_{\text{Pulse}} \cdot N_{\text{Rx}}. \tag{5}$$

Our model consists of L measurement cycles, where each cycle is made up of N_{Pulse} pulses. The baseband signal of cycle l is

$$\underline{X}(l) = \sqrt{\underline{\rho}^{\text{virt}}} \odot \exp(j\underline{t}^{\text{virt}}\omega) \odot \exp(j\underline{d}^{\text{virt}}u)s(l) + \underline{N}(l), \quad l = 1, \ldots, L. \tag{6}$$

Here, $s(l) \in \mathbb{C}$ is the unknown, deterministic complex target signal and $\underline{N}(l)$ is the noise.

$$\underline{\rho}^{\text{virt}} = \underline{\rho} \otimes \underline{1}_{N_{\text{Rx}}} \tag{7}$$

where ρ_i is the transmitted energy of pulse i.

$$\underline{t}^{\text{virt}} = \underline{t} \otimes \underline{1}_{N_{\text{Rx}}} \tag{8}$$

where $\underline{t}$ denotes the transmit time instances and ω is the target's angular Doppler frequency. The term $\exp(j\underline{t}^{\text{virt}}\omega)$ describes the phase change due to the Doppler effect caused by the moving target. We define a new steering vector

$$\underline{b}(u, \omega) = \sqrt{\underline{\rho}^{\text{virt}}} \odot \exp(j\underline{t}^{\text{virt}}\omega) \odot \exp(j\underline{d}^{\text{virt}}u) \tag{9}$$

to write the baseband signal as

$$\underline{X}(l) = \underline{b}(u, \omega)s(l) + \underline{N}(l). \tag{10}$$

For two targets, the baseband signal is the sum of the baseband signals of both targets. Hence

$$\underline{X}(l) = \mathbf{B}(\underline{u}, \underline{\omega})\underline{s}(l) + \underline{N}(l), \tag{11}$$

$$\mathbf{B} = [\underline{b}(u_1, \omega_1) \ \underline{b}(u_2, \omega_2)] \tag{12}$$

with $\underline{u} = [u_1, u_2]^T$, $\underline{\omega} = [\omega_1, \omega_2]^T$ and $\underline{s}(l) = [s_1(l), s_2(l)]^T$.

We make the following assumptions:

- $\underline{N}(l)$ is circular complex Gaussian with zero mean, spatially and temporally uncorrelated with $\text{E}\left(\underline{N}(l)\ \underline{N}^H(m)\right) = \delta_{l,m}\ \sigma^2\mathbf{I}$.
- The targets' distances to the MIMO radar are much larger than the aperture of the radar such that we have a far field situation. Hence the radar receives a plane wave for every target and the radar cross sections as well as the DOAs $\underline{\vartheta}$ of the targets are the same for all antennas.
- The DOAs $\underline{\vartheta}$ do not change significantly during the L measurement cycles, i.e. the change is much smaller than the DOA accuracy of the radar and can thus be ignored.
- The targets move with constant relative radial velocity during the L measurement cycles. Hence their Doppler frequencies $\underline{\omega}$ are constant.

The unknown quantities to be estimated from $\underline{X}(l)$, $1 \leq l \leq L$, are denoted by

$$\underline{\Theta} = [u_1, \omega_1, u_2, \omega_2, \underline{s}(1), \ldots, \underline{s}(L), \sigma^2]^T. \tag{13}$$

III. Cramer-Rao Bound

A. Derivation

In our model, the unknown parameter vector is given in (13). We want to obtain the part $\text{CRB}_{(\underline{\Theta}^{(1)}, \underline{\Theta}^{(2)})}$ of the CRB corresponding to the parameters $\underline{\Theta}^{(1)}, \underline{\Theta}^{(2)}$ with $\underline{\Theta}^{(i)} = [u_i, \omega_i]^T$, $i = 1, 2$. Hence we have to compute the FIM $\mathbf{J}$ of the whole parameter vector $\underline{\Theta}$ and then its inverse $\mathbf{J}^{-1}$. After that, we take that 4×4 block corresponding to $\underline{\Theta}^{(1)}, \underline{\Theta}^{(2)}$. This part is $\text{CRB}_{(\underline{\Theta}^{(1)}, \underline{\Theta}^{(2)})}$. Following [5],

$$\text{CRB}^{-1}_{(\underline{\Theta}^{(1)}, \underline{\Theta}^{(2)})} = \frac{2\ L}{\sigma^2}\text{Re}\left[\mathbf{C} \odot \left(\mathbf{S}^T \otimes \mathbf{1}_{2\times 2}\right)\right] \in \mathbb{R}^{4\times 4} \tag{14}$$

where

$$\mathbf{C} = \mathbf{D}^H \mathbf{P}_\mathbf{B}^\perp \mathbf{D} \in \mathbb{C}^{4\times 4}, \tag{15}$$

$$\mathbf{D} = [\mathbf{D}_1, \mathbf{D}_2] \in \mathbb{C}^{N_\text{virt}\times 4}, \tag{16}$$

$$\mathbf{D}_i = \left[\frac{\partial \underline{b}(u_i,\omega_i)}{\partial u_i}, \frac{\partial \underline{b}(u_i,\omega_i)}{\partial \omega_i}\right] \in \mathbb{C}^{N_\text{virt}\times 2}, \tag{17}$$

$$\mathbf{P}_\mathbf{B}^\perp = \mathbf{I} - \mathbf{P}_\mathbf{B} \in \mathbb{C}^{N_\text{virt}\times N_\text{virt}}, \tag{18}$$

$$\mathbf{P}_\mathbf{B} = \mathbf{B}(\mathbf{B}^H\ \mathbf{B})^{-1}\mathbf{B}^H \in \mathbb{C}^{N_\text{virt}\times N_\text{virt}}, \tag{19}$$

$$\mathbf{S} = \frac{1}{L}\sum_{l=1}^{L} \underline{s}(l)\underline{s}^H(l) = \begin{bmatrix} \sigma_{s,1}^2 & c_s \\ c_s^* & \sigma_{s,2}^2 \end{bmatrix} \in \mathbb{C}^{2\times 2}. \tag{20}$$

After some lengthy calculations, this results in

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 - \mathbf{C}_3 \tag{21}$$

$$\mathbf{C}_1 = \begin{bmatrix} \rho_\text{sum}^\text{virt}\,\mathrm{Cov}^\mathrm{S}(\mathbf{V}_1) & b_{\Delta,\text{sum}}^*\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_1, \underline{b}_{\Delta,Rx}^*) \\ b_{\Delta,\text{sum}}\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_1, \underline{b}_{\Delta,Rx}) & \rho_\text{sum}^\text{virt}\,\mathrm{Cov}^\mathrm{S}(\mathbf{V}_1) \end{bmatrix} \tag{22}$$

$$\mathbf{C}_2 = \begin{bmatrix} \rho_\text{sum}^\text{virt}\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{\rho}) & b_{\Delta,\text{sum}}^*\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{b}_{\Delta,Tx}^*) \\ b_{\Delta,\text{sum}}\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{b}_{\Delta,Tx}) & \rho_\text{sum}^\text{virt}\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{\rho}) \end{bmatrix} \tag{23}$$

$$\mathbf{C}_3 = \frac{\rho_\text{sum}^\text{virt}}{{\rho_\text{sum}^\text{virt}}^2 - |b_{\Delta,\text{sum}}|^2} \cdot \begin{bmatrix} |b_{\Delta,\text{sum}}|^2 \underline{m}^* \underline{m}^T & -\rho_\text{sum}^\text{virt} b_{\Delta,\text{sum}}^* \underline{m}^* \underline{m}^H \\ -\rho_\text{sum}^\text{virt} b_{\Delta,\text{sum}} \underline{m}\ \underline{m}^T & |b_{\Delta,\text{sum}}|^2 \underline{m}\ \underline{m}^H \end{bmatrix} \tag{24}$$

with

$$\rho_\text{sum}^\text{virt} = \underline{1}^T \underline{\rho}^\text{virt} = N_\text{Rx} \underline{1}^T \underline{\rho}, \tag{25}$$

$$\Delta u = u_1 - u_2, \quad \Delta\omega = \omega_1 - \omega_2, \tag{26}$$

$$\underline{b}_{\Delta,Tx} = \underline{\rho} \odot \exp(j\underline{t}\Delta\omega) \odot \exp(\ j\underline{d}^\text{Pulse}\Delta u), \tag{27}$$

$$\underline{b}_{\Delta,Rx} = \exp(j\underline{d}^\text{Rx}\Delta u), \tag{28}$$

$$\underline{b}_\Delta = \underline{b}_{\Delta,Tx} \otimes \underline{b}_{\Delta,Rx}, \tag{29}$$

$$b_{\Delta,\text{sum}} = \underline{1}^T \underline{b}_\Delta = \left(\underline{1}^T \underline{b}_{\Delta,Tx}\right)\left(\underline{1}^T \underline{b}_{\Delta,Rx}\right), \tag{30}$$

$$\underline{m} = [\mathrm{E}^\mathrm{S}(\mathbf{V}_1) - \mathrm{E}^\mathrm{WS}(\mathbf{V}_1, \underline{b}_{\Delta,Rx}) + \mathrm{E}^\mathrm{WS}(\mathbf{V}_2, \underline{\rho}) - \mathrm{E}^\mathrm{WS}(\mathbf{V}_2, \underline{b}_{\Delta,Tx})]^T, \tag{31}$$

$$\mathbf{V}_1 = [\underline{d}^\text{Rx}, \underline{0}], \quad \mathbf{V}_2 = [\underline{d}^\text{Pulse}, \underline{t}]. \tag{32}$$

Here we use the following definitions for two matrices $\mathbf{X} \in \mathbb{C}^{K\times N_1}$, $\mathbf{Y} \in \mathbb{C}^{K\times N_2}$ and one weight vector $\underline{w} \in \mathbb{C}^K$:

- weighted sample mean

$$\mathrm{E}^\mathrm{WS}(\mathbf{X}, \underline{w}) = \frac{1}{\underline{1}^T \underline{w}} \underline{w}^T \mathbf{X} \in \mathbb{C}^{1\times N_1} \tag{33}$$

- weighted sample cross-correlation

$$\mathrm{Corr}^\mathrm{WS}(\mathbf{X}, \mathbf{Y}, \underline{w}) = \frac{1}{\underline{1}^T \underline{w}} \mathbf{Y}^H \operatorname{diag}(\underline{w})\ \mathbf{X} \in \mathbb{C}^{N_2\times N_1} \tag{34}$$

- weighted sample cross-covariance

$$\mathrm{Cov}^\mathrm{WS}(\mathbf{X}, \mathbf{Y}, \underline{w}) = \mathrm{Corr}^\mathrm{WS}\left(\mathbf{X} - \underline{1}\ \mathrm{E}^\mathrm{WS}(\mathbf{X}, \underline{w})\ , \mathbf{Y} - \underline{1}\ \mathrm{E}^\mathrm{WS}(\mathbf{Y}, \underline{w}^*)\ , \underline{w}\right) \in \mathbb{C}^{N_2\times N_1} \tag{35}$$

- weighted sample covariance

$$\mathrm{Cov}^\mathrm{WS}(\mathbf{X}, \underline{w}) = \mathrm{Cov}^\mathrm{WS}(\mathbf{X}, \mathbf{X}, \underline{w}) \in \mathbb{C}^{N_1\times N_1} \tag{36}$$

If $\underline{w} \propto \underline{1}_K$, all above defined weighted sample mean, cross-correlation, cross-covariance and covariance simplify to simple sample mean $\mathrm{E}^\mathrm{S}(\mathbf{X})$, sample cross-correlation $\mathrm{Corr}^\mathrm{S}(\mathbf{X}, \mathbf{Y})$, sample cross-covariance $\mathrm{Cov}^\mathrm{S}(\mathbf{X}, \mathbf{Y})$ and sample covariance $\mathrm{Cov}^\mathrm{S}(\mathbf{X})$, respectively. In the special case of a vector $\underline{x} \in \mathbb{C}^K$, we write the weighted sample variance $\mathrm{Var}^\mathrm{WS}(\underline{x}, \underline{w})$ and the sample variance $\mathrm{Var}^\mathrm{S}(\underline{x})$ instead of $\mathrm{Cov}^\mathrm{WS}(\underline{x}, \underline{w})$ and $\mathrm{Cov}^\mathrm{S}(\underline{x})$, respectively.

B. Notes on the CRB

The matrix $\mathbf{C}$ consists of 3 terms. $\mathbf{C}_1$ is determined by the Rx-array $\underline{d}^\text{Rx}$. $\mathrm{Cov}^\mathrm{S}(\mathbf{V}_1) = \begin{bmatrix} \mathrm{Var}^\mathrm{S}(\underline{d}^\text{Rx}) & 0 \\ 0 & 0 \end{bmatrix}$ describes the influence of $\underline{d}^\text{Rx}$ on the estimation of u_1 and u_2. The 0 entries show that $\mathrm{Cov}^\mathrm{S}(\mathbf{V}_1)$ does not contain any information about $\underline{\omega}$. $\mathrm{Var}^\mathrm{S}(\underline{d}^\text{Rx})$ is the same expression as the one which appears in the CRB for a SIMO radar for the one target DOA estimation [6]. In $\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_1, \underline{b}_{\Delta,Rx}) = \begin{bmatrix} \mathrm{Var}^\mathrm{WS}(\underline{d}^\text{Rx}, \underline{b}_{\Delta,Rx}) & 0 \\ 0 & 0 \end{bmatrix}$, there is again no information about $\underline{\omega}$. $\mathrm{Var}^\mathrm{WS}(\underline{d}^\text{Rx}, \underline{b}_{\Delta,Rx})$ expresses the coupling between the two electrical angles. The coupling depends on the angle separation Δu of the two targets.

$\mathbf{C}_2$ is determined by the TDM sequence, namely $\underline{d}^\text{Pulse}$ and $\underline{t}$. $\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{\rho}) = \begin{bmatrix} \mathrm{Var}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{\rho}) & \mathrm{Cov}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{t}, \underline{\rho}) \\ \mathrm{Cov}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{t}, \underline{\rho}) & \mathrm{Var}^\mathrm{WS}(\underline{t}, \underline{\rho}) \end{bmatrix}$ is analog to the one-target DOA estimation case [4]: $\mathrm{Var}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{\rho})$ describes the amount of information on u_1, u_2 due to the Tx antennas. $\mathrm{Var}^\mathrm{WS}(\underline{t}, \underline{\rho})$ contains the information for estimating ω_1 and ω_2 due to the different Tx times. $\mathrm{Cov}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{t}, \underline{\rho})$ is the coupling between u and ω of the same target. If $\underline{d}^\text{Pulse}$, $\underline{t}$ and $\underline{\rho}$ are chosen in a suitable way, this coupling can be reduced to 0 [4]. $b_{\Delta,\text{sum}}\,\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{b}_{\Delta,Tx})$ expresses the coupling between the two DOAs and two Doppler frequencies. Since $\mathrm{Cov}^\mathrm{WS}(\mathbf{V}_2, \underline{b}_{\Delta,Tx}) = \begin{bmatrix} \mathrm{Var}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{b}_{\Delta,Tx}) & \mathrm{Cov}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{t}, \underline{b}_{\Delta,Tx}) \\ \mathrm{Cov}^\mathrm{WS}(\underline{d}^\text{Pulse}, \underline{t}, \underline{b}_{\Delta,Tx}) & \mathrm{Var}^\mathrm{WS}(\underline{t}, \underline{b}_{\Delta,Tx}) \end{bmatrix}$, Δu as well as $\Delta\omega$ affect the amount of coupling between the two targets' DOAs and Doppler frequencies.

The matrix $\mathbf{C}_3$ can be interpreted in the following way: $\mathbf{C}_3$ origins from $\mathbf{D}^H \mathbf{P}_\mathbf{B} \mathbf{D}$. $\mathbf{P}_\mathbf{B}$ is the projector onto the subspace W spanned by the column vectors of $\mathbf{B}$, namely the steering vectors $\underline{b}(u_1, \omega_1), \underline{b}(u_2, \omega_2)$. $\mathbf{D}$ contains the derivatives of the steering vectors. Those vectors are perpendicular to the subspace W iff $\mathbf{D}^H \mathbf{P}_\mathbf{B} \mathbf{D} = \mathbf{0}$. It can be shown, without loss of generality, that this is equivalent to $\mathbf{C}_3 = \mathbf{0}$. Hence, roughly speaking, $\mathbf{C}_3$ denotes the part of the derivatives of the steering vectors which is in W. The smaller the diagonal elements of $\mathbf{C}_3$, the more information is contained in the diagonal elements of $\mathbf{C}$. This is in accordance to the geometrical interpretation: the radar system is sensitive if the change of the steering vectors due to a change of parameters is perpendicular to the steering vectors. Note that $\mathbf{C}_3$ has mixed contributions from both the Rx and Tx array and contains couplings of both targets' DOAs and Doppler frequencies.

Note that $\mathrm{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})}$ depends not only on the position of the Tx antennas, but also on the TDM scheme described by the transmitting order $\underline{d}^{\text{Pulse}}$ and transmitting times $\underline{t}$.

It can be shown that $\mathrm{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})}$ is a function of $\Delta u, \Delta\omega, \alpha$, where α is the phase of c_s, i.e. $c_s = |c_s| \exp(j\alpha)$

$$\mathrm{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})} = \mathrm{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})}(\Delta u, \Delta\omega, \alpha). \tag{37}$$

Hence it does not depend on the absolute values of u_i, ω_i, but only on their difference. Similarly, it does not depend on the phases of $s_i(l)$ but only on α. For one cycle with $L = 1$, α is the difference of the phases of s_2 and s_1.

IV. Decoupling of Electrical Angle and Doppler Frequency

In [4] the DOA estimation of one moving target was investigated. It was shown that TDM schemes can be found such that the CRB has the same value as for a stationary target. Similarly, we are interested here in TDM schemes which lead to a decoupling of the electrical angles and Doppler frequencies in the CRB. We consider the worst case $\Delta\omega = 0$, where the two targets can be distinguished only by their DOAs. Note, since the estimator does not know a priori that $\Delta\omega = 0$, both Doppler frequencies ω_1 and ω_2 have to be estimated.

Theorem 1: Let $\Delta\omega = 0$. The elements of $\mathrm{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})}$ corresponding to u_1, u_2 decouple from the elements corresponding to ω_1, ω_2 if the TDM scheme satisfies

$$\mathrm{E}^{\mathrm{WS}}(\underline{t}^{(k)}, \underline{\rho}^{(k)}) = \mathrm{E}^{\mathrm{WS}}(\underline{t}^{(l)}, \underline{\rho}^{(l)}) \ \forall k, l \in \{1, \ldots, N_{\mathrm{Tx}}\}. \tag{38}$$

Here $\underline{t}^{(k)}$ are the time instances when the k-th Tx antenna transmits and $\underline{\rho}^{(k)}$ are the corresponding energies of the transmitted pulses. From (38), it follows

$$\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\text{Pulse}}, \underline{t}, \underline{\rho}) = 0. \tag{39}$$

For two Tx antennas, (38) and (39) are equivalent. The proof is given in the Appendix.

The decoupling means, after a permutation of the parameters $u_1, u_2, \omega_1, \omega_2$, the CRB has a block diagonal structure. We denote the first 2×2 block for $\underline{u} = [u_1, u_2]^T$ by $\mathrm{CRB}_{\underline{u}}$. Moreover, due to the decoupling, $\mathrm{CRB}_{\underline{u}}$ is equal to the CRB when the Doppler frequencies are known and do not have to be estimated

$$\mathrm{CRB}_{\underline{u}} = \mathrm{CRB}_{\underline{u}, \text{ known } \underline{\omega}}. \tag{40}$$

Hence, if a TDM scheme satisfying (38) is chosen, a DOA accuracy can be achieved which is as good as in the stationary case, i.e. the complete virtual aperture can be used for the estimation of the electrical angles despite the unknown Doppler frequencies.

Note that for $\rho \propto \underline{1}$, it was shown in [4] that (39) is the condition for the decoupling of electrical angle and Doppler frequency in the one target case. In the two target case, if $\underline{\rho} \propto \underline{1}$, (38) states that the mean value of the transmitting times must be the same for all Tx antennas.

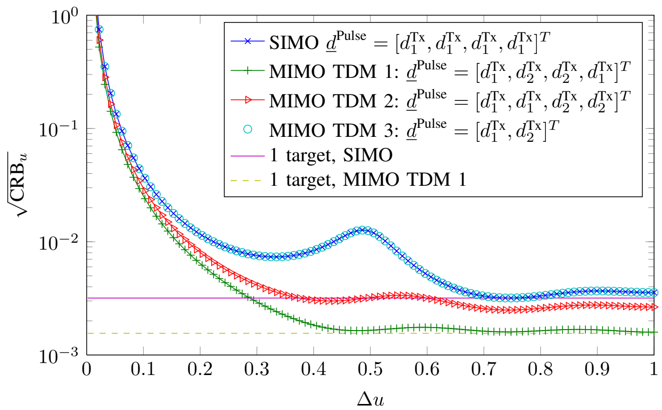

Fig. 2. $\sqrt{\mathrm{CRB}_u}$ for DOA estimation of two and one moving target

In Fig. 2, the CRB of the electrical angle CRB_u is depicted for different TDM schemes and for varying Δu. The radar system has the following parameters: the number of measurement cycles is $L = 1$. The Rx array is an 4-element ULA with half wavelength spacing, $\underline{d}^{\text{Rx}} = \pi \cdot [-1.5, -0.5, 0.5, 1.5]^T$. The Tx antenna positions are $\underline{d}^{\text{Tx}} = \pi \cdot [-2, 2]^T$. Both targets have the same signal strength $s_1 = s_2 = 1$ and we set $\Delta\omega = 0$. The SNR, defined as $\mathrm{SNR} = |s_1|^2/\sigma^2$, is 30dB. The transmitted energy per pulse is $\underline{\rho} = \frac{1}{N_{\text{Pulse}}}\underline{1}$ ensuring a constant total transmitted energy independent of the number of pulses. The transmit times are $\underline{t} = [0, \ldots, N_{\text{Pulse}}-1]^T$. Since both targets have the same signal strength and $\Delta\omega = 0$, the CRBs of their electrical angles are equal, which we denote by CRB_u in the following: $\mathrm{CRB}_u = \left[\mathrm{CRB}_{\underline{u}}\right]_{1,1} = \left[\mathrm{CRB}_{\underline{u}}\right]_{2,2}$. Three different TDM schemes are considered: TDM scheme 1, for which the electrical angles decouple from the Doppler frequencies, and TDM scheme 2 and 3 which do not satisfy condition (38). Fig. 2 shows that for $\Delta\omega = 0$ in general $\mathrm{CRB}_{u,\text{TDM 1}} \leq \mathrm{CRB}_{u,\text{TDM 2}}$. For large Δu, CRB_u for the two-target case reaches CRB_u for one moving target, which was computed in [4]. TDM scheme 3 achieves the same CRB_u as using only one transmitter, i.e. the SIMO case, because the Doppler frequencies have to be estimated as well. Since both Tx transmit only once, it cannot be distinguished if the phase difference of the Tx elements is due to the Doppler or due to the angle. This is the same observation as in the one-target case [4]. Hence, in order to achieve a small CRB, it is crucial to choose a good TDM scheme.

V. Numerical Simulation

In the following numerical simulations are presented to verify the theoretical computations. We determine the root mean square error (RMSE) of the maximum likelihood (ML) estimator for the electrical angle u_1 and u_2 of the 1. and 2. target, respectively, by Monte Carlo simulations. The parameters are the same as in Fig. 2, but with $\mathrm{SNR} = 25$dB. We consider TDM scheme 1 with $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, d_2^{\text{Tx}}, d_2^{\text{Tx}}, d_1^{\text{Tx}}]^T$. 3000 Monte Carlo simulations are done for every value of Δu. Since the ML estimator is nonlinear in $u_1, u_2, \omega_1, \omega_2$, the ML estimates are computed by a 4 dimensional grid search, in

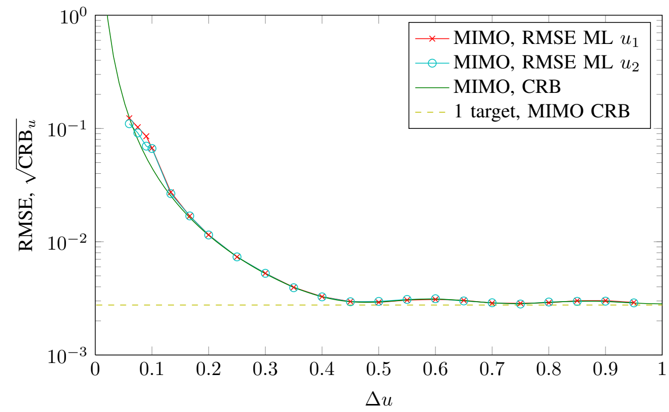

Fig. 3. DOA estimation of two moving targets with TDM scheme 1, $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, d_2^{\text{Tx}}, d_2^{\text{Tx}}, d_1^{\text{Tx}}]^T$. Comparison of $\sqrt{\text{CRB}_u}$ and the RMSE of the maximum likelihood estimator for u_1 and u_2.

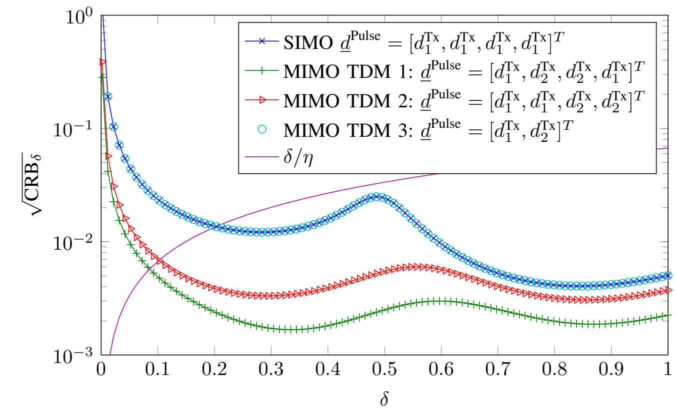

Fig. 4. Angular resolution limit for DOA estimation of two moving targets: it is obtained by the intersection of δ/η and $\sqrt{\text{CRB}_\delta(\delta)}$

order to find the global maximum of the likelihood, followed by a local optimization. Since the grid search is computational expensive, for $\Delta u \geq 0.2$, the ML estimates are computed by a local optimization starting at the true value. This is possible, since the ML estimator already achieves the CRB, as shown in Fig. 3, and hence global errors are negligible. In this case the number of Monte Carlo simulations is 10000.

Fig. 3 shows, that for $\Delta u \geq 0.13$ the ML estimator reaches the CRB. For large Δu the 1 target CRB is achieved. For $\Delta u < 0.13$, the ML deviates from the CRB due to global errors. If Δu is very small, the RMSE of the u_1- and u_2-estimator differ. The reason is that the smaller estimated electrical angle is always assigned to the 1. target and hence the targets' angles can be swapped. Note that for small Δu the RMSE could be even smaller than the CRB, since the CRB does not incorporate the periodicity of the electrical angle.

VI. Angular Resolution Limit

An important performance criterion of a radar system is its angular resolution limit, i.e. the angular separation at which two targets can be distinguished. There are different possibilities to define the resolution, see a survey in [7]. In [8] the resolution is defined by statistical means: a detection based approach is used to define the resolution limit and the relation to the CRB is shown. The authors of [9] extend the resolution definition to multiple targets and multiple parameters of interest. Hence we can use the two-target CRB to define the angular resolution limit analytically. We use the definition of the resolution limit given in [9]. The unknown parameters of the signal model are given in (13). Instead of estimating $u_1, u_2, \omega_1, \omega_2$, we can also rewrite the model and estimate $u_1, \Delta u, \omega_1, \Delta\omega$. Hence the unknown parameters are

$$\underline{\Theta} = [u_1, \omega_1, \Delta u, \Delta\omega, \underline{s}(1), \ldots, \underline{s}(L), \sigma^2]^T. \quad (41)$$

For the computation of the resolution limit, the parameters of interest are $\Delta u, \Delta\omega$ and the nuisance parameters are $[u_1, \omega_1, \underline{s}(1), \ldots, \underline{s}(L), \sigma^2]^T$. We investigate the case with $\Delta\omega = 0$ and equal signal strengths $s_1 = s_2$. Then, using (37) and due to symmetry, $\text{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})}$ depends on $|\Delta u|$. The angular resolution limit $\delta = |\Delta u|$ is given as the solution of [9]

$$\delta = \eta\sqrt{\text{CRB}_\delta(\delta)}. \quad (42)$$

Here, η is a factor which depends on the probability of false alarm P_{FA} and probability of detection P_{D}

$$\eta = \eta(P_{\text{FA}}, P_{\text{D}}) \quad (43)$$

and CRB_δ is the CRB of the parameter $\delta = |\Delta u|$. It can be computed by using the rule of transformation of parameters [10] and is given by [9]

$$\text{CRB}_\delta = \left[\text{CRB}_{\underline{u}}\right]_{1,1} + \left[\text{CRB}_{\underline{u}}\right]_{2,2} - 2\left[\text{CRB}_{\underline{u}}\right]_{1,2}. \quad (44)$$

Using (42) the resolution limit can be obtained by the intersection of δ/η and $\sqrt{\text{CRB}_\delta}$. We set $P_{\text{FA}} = 0.01$ and $P_{\text{D}} = 0.9$ which result in $\eta \approx 14.9$.

δ/η and $\sqrt{\text{CRB}_\delta}$ are depicted in Fig. 4 for different TDM schemes. The parameters of the radar system are the same as in Fig. 2. The TDM scheme 1 with $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, d_2^{\text{Tx}}, d_2^{\text{Tx}}, d_1^{\text{Tx}}]^T$ achieves the best resolution limit. In this case, $\underline{d}^{\text{Pulse}}$ satisfies the condition of Th. 1 which yields a decoupling of the electrical angles and Doppler frequencies. TDM scheme 2 does not satisfy the condition of Th. 1. It achieves a worse resolution limit. TDM scheme 3 achieves the same resolution limit as in the SIMO, i.e. using only one transmitter. This is similar to Fig. 2 and explained there.

Hence the resolution of a TDM MIMO radar depends crucially on the chosen TDM scheme and can be significantly improved compared to a SIMO radar by optimizing the TDM scheme.

VII. Conclusion

We derived the CRB for the DOA estimation of two moving targets for a MIMO radar using TDM. Conditions for TDM schemes were derived such that the CRBs for the DOAs and Doppler frequencies are decoupled. The CRB was used to define the angular resolution limit. We have shown that the angular resolution limit depends on the chosen TDM scheme of the MIMO radar and can be better than that of the SIMO radar, despite the unknown Doppler frequencies.

VIII. Appendix: Proof of Th. 1

Th. 1 states that the elements marked with $\times$ are 0

$$\mathrm{CRB}_{(\underline{\Theta}^{(1)},\underline{\Theta}^{(2)})} = \begin{bmatrix} . & \times & . & \times \\ \times & . & \times & . \\ . & \times & . & \times \\ \times & . & \times & . \end{bmatrix} \tag{45}$$

if $\Delta\omega = 0$ and (38) is satisfied. Due to (14) it is sufficient to show that the corresponding elements in $\mathbf{C}$ are 0. It can be verified that $\mathbf{C}$ is hermitian, i.e. $\mathbf{C} = \mathbf{C}^H$. Hence it is sufficient to proof that $[\mathbf{C}]_{1,2} = [\mathbf{C}]_{1,4} = [\mathbf{C}]_{2,3} = [\mathbf{C}]_{3,4} = 0$. Using (21), we show that the corresponding elements of $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ are 0.

We can verify that

$$\begin{aligned} &\mathrm{E}^{\mathrm{WS}}(\underline{t}^{(k)}, \underline{\rho}^{(k)}) = \mathrm{E}^{\mathrm{WS}}(\underline{t}^{(l)}, \underline{\rho}^{(l)}) \ \forall k,l \in \{1,\dots,N_{\mathrm{Tx}}\} \\ &\Rightarrow \mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0. \end{aligned} \tag{46}$$

Hence for TDM schemes satisfying (38) follows $\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0$.

Proof of $[\mathbf{C}]_{1,2} = 0$:

$$[\mathbf{C}_1]_{1,2} \propto \left[\mathrm{Cov}^{\mathrm{S}}(\mathbf{V}_1)\right]_{1,2} = \mathrm{Cov}^{\mathrm{S}}(\underline{d}^{\mathrm{Rx}}, \underline{0}) = 0 \tag{47}$$

$$\begin{aligned} [\mathbf{C}_2]_{1,2} &\propto \left[\mathrm{Cov}^{\mathrm{WS}}(\mathbf{V}_2, \underline{\rho})\right]_{1,2} \\ &= \mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0 \end{aligned} \tag{48}$$

$$[\mathbf{C}_3]_{1,2} \propto \left[\underline{m}^* \underline{m}^T\right]_{1,2} = m_1^* m_2 \tag{49}$$

$$m_2 = \mathrm{E}^{\mathrm{WS}}(\underline{t}, \underline{\rho}) - \mathrm{E}^{\mathrm{WS}}(\underline{t}, \underline{b}_{\Delta,Tx}) \tag{50}$$

With $\Delta\omega = 0$ follows

$$\underline{b}_{\Delta,Tx} = \underline{\rho} \odot \exp(j\underline{d}^{\mathrm{Pulse}} \Delta u). \tag{51}$$

Using (38) we can show after some computations that

$$\mathrm{E}^{\mathrm{WS}}(\underline{t}, \underline{\rho}) = \mathrm{E}^{\mathrm{WS}}(\underline{t}, \exp(j\underline{d}^{\mathrm{Pulse}} \Delta u) \odot \underline{\rho}). \tag{52}$$

Hence

$$m_2 = 0 \tag{53}$$

and therefore $[\mathbf{C}_3]_{1,2} = 0$.

Proof of $[\mathbf{C}]_{3,4} = 0$: this follows directly from the former computations:

$$[\mathbf{C}_1]_{3,4} = [\mathbf{C}_1]_{1,2} = 0 \tag{54}$$

$$[\mathbf{C}_2]_{3,4} = [\mathbf{C}_2]_{1,2} = 0 \tag{55}$$

and using (53) results in

$$[\mathbf{C}_3]_{3,4} \propto \left[\underline{m}\ \underline{m}^H\right]_{1,2} = m_1 m_2^* = 0. \tag{56}$$

Proof of $[\mathbf{C}]_{1,4} = 0$:

$$\begin{aligned} [\mathbf{C}_1]_{1,4} &\propto \left[\mathrm{Cov}^{\mathrm{WS}}(\mathbf{V}_1, \underline{b}^*_{\Delta,Rx})\right]_{1,2} \\ &= \mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Rx}}, \underline{0}, \underline{b}^*_{\Delta,Rx}) = 0 \end{aligned} \tag{57}$$

$$\begin{aligned} [\mathbf{C}_2]_{1,4} &\propto \left[\mathrm{Cov}^{\mathrm{WS}}(\mathbf{V}_2, \underline{b}^*_{\Delta,Tx})\right]_{1,2} \\ &= \mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{b}^*_{\Delta,Tx}) \end{aligned} \tag{58}$$

Using (51) and (38) computations yield

$$\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{b}^*_{\Delta,Tx}) = 0 \tag{59}$$

and hence $[\mathbf{C}_2]_{1,4} = 0$. With (53) follows

$$[\mathbf{C}_3]_{1,4} \propto \left[\underline{m}^* \underline{m}^H\right]_{1,2} = m_1^* m_2^* = 0. \tag{60}$$

Proof of $[\mathbf{C}]_{2,3} = 0$:

$$\begin{aligned} [\mathbf{C}_1]_{2,3} &\propto \left[\mathrm{Cov}^{\mathrm{WS}}(\mathbf{V}_1, \underline{b}^*_{\Delta,Rx})\right]_{2,1} \\ &= \mathrm{Cov}^{\mathrm{WS}}(\underline{0}, \underline{d}^{\mathrm{Rx}}, \underline{b}^*_{\Delta,Rx}) = 0 \end{aligned} \tag{61}$$

Using (59) gives

$$\begin{aligned} [\mathbf{C}_2]_{2,3} &\propto \left[\mathrm{Cov}^{\mathrm{WS}}(\mathbf{V}_2, \underline{b}^*_{\Delta,Tx})\right]_{2,1} \\ &= \mathrm{Cov}^{\mathrm{WS}}(\underline{t}, \underline{d}^{\mathrm{Pulse}}, \underline{b}^*_{\Delta,Tx}) \\ &= \mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{b}^*_{\Delta,Tx}) = 0. \end{aligned} \tag{62}$$

And with (53) follows

$$[\mathbf{C}_3]_{2,3} \propto \left[\underline{m}^* \underline{m}^H\right]_{2,1} = m_2^* m_1^* = 0. \tag{63}$$

References

[1] I. Bekkerman and J. Tabrikian, "Target Detection and Localization Using MIMO Radars and Sonars," *IEEE Transactions on Signal Processing*, vol. 54, no. 10, pp. 3873–3883, 2006.

[2] Jian Li and P. Stoica, "MIMO Radar with Colocated Antennas," *IEEE Signal Processing Magazine*, vol. 24, no. 5, pp. 106–114, 2007.

[3] Jian Li, Luzhou Xu, P. Stoica, K. W. Forsythe, and D. W. Bliss, "Range Compression and Waveform Optimization for MIMO Radar: A Cramer-Rao Bound Based Study," *IEEE Transactions on Signal Processing*, vol. 56, no. 1, pp. 218–232, 2008.

[4] K. Rambach and B. Yang, "Colocated MIMO Radar: Cramer-Rao Bound and Optimal Time Division Multiplexing for DOA Estimation of Moving Targets," in *Proc. IEEE Int. Conf. Acoustics, Speech and Signal Processing (ICASSP)*, 2013.

[5] S. F. Yau and Y. Bresler, "A compact Cramer-Rao bound expression for parametric estimation of superimposed signals," *IEEE Transactions on Signal Processing*, vol. 40, no. 5, pp. 1226–1230, 1992.

[6] F. Athley, "Threshold region performance of maximum likelihood direction of arrival estimators," *IEEE Transactions on Signal Processing*, vol. 53, no. 4, pp. 1359–1373, 2005.

[7] A. J. den Dekker and A. van den Bos, "Resolution: a survey," *J. Opt. Soc. Am. A*, vol. 14, pp. 547–557, 1997.

[8] Z. Liu and A. Nehorai, "Statistical Angular Resolution Limit for Point Sources," *IEEE Transactions on Signal Processing*, vol. 55, no. 11, pp. 5521–5527, 2007.

[9] M.N. El Korso, R. Boyer, A. Renaux, and S. Marcos, "A GLRT-based framework for the Multidimensional Statistical Resolution Limit," in *Statistical Signal Processing Workshop (SSP), 2011 IEEE*, 2011, pp. 453–456.

[10] S. M. Kay, *Estimation Theroy*, vol. I of *Fundamentals of Statistical Signal Processing*, Prentice Hall PTR, Upper Saddle River, NJ, 1993.