Optimal Time Division Multiplexing Schemes for DOA Estimation of a Moving Target Using a Colocated MIMO Radar

Kilian Rambach, Markus Vogel and Bin Yang

Institute of Signal Processing and System Theory
University of Stuttgart, Germany
Email: kilian.rambach@iss.uni-stuttgart.de, markus.karl.vogel@googlemail.com, bin.yang@iss.uni-stuttgart.de

***Abstract*—A Multiple-Input-Multiple-Output (MIMO) radar can achieve a higher accuracy in direction of arrival (DOA) estimation compared to a corresponding Single-Input-Multiple-Output (SIMO) radar due to its larger virtual aperture. If the target is moving relative to the radar, an additional phase shift is introduced into the baseband signal because of the Doppler effect. Hence the Doppler frequency has to be estimated in addition to the DOA. In general, this decreases the DOA estimation accuracy. We investigate MIMO radars using time division multiplexing (TDM). We derive the Cramer-Rao bound (CRB) for a moving target and a general TDM scheme and compare it to other radar systems. Moreover we derive optimal TDM schemes which minimize the CRB under different constraints.**

***Index Terms*—MIMO radar, DOA estimation, Time Division Multiplexing, Cramer Rao Bound, CRB**

I. Introduction

Compared to a conventional SIMO radar, MIMO radars have several advantages, e. g. a flexible transmit beampattern design and a high DOA accuracy [1], [2]. We are interested in the DOA accuracy. The CRB is a lower bound on the covariance matrix of any unbiased estimator. Thus, it can be used to study the maximum accuracy of a radar system. The CRB has been computed for different systems. In [1], [3] the authors investigated DOA estimation with a MIMO radar for stationary targets. Here we consider a TDM-MIMO radar and study the DOA estimation of a non-stationary target, i. e. the target moves relative to the radar. To estimate the DOA, the phases of the complex baseband signal are processed. Since the target is moving, the Doppler effect causes additional phase shifts. Therefore, the Doppler frequency has to be estimated in addition. In general, this decreases the accuracy of the DOA estimation. In [4], [5] the DOA estimation of one and two moving targets using a TDM-MIMO radar has been investigated. It was shown that the DOA accuracy depends on the position as well as on the transmission sequence of the Tx antennas. We extend the model of [4] to pulses with different transmission powers and durations and derive the CRB. We compare the CRB to that of a SIMO radar and a MIMO radar with a stationary target. Moreover, we present conditions on the TDM schemes such that the CRB is minimized under some typical constraints, like the maximum transmission power and physical antenna aperture. Numerical simulations are presented, which confirm the analytical findings.

We use the following notations in the paper: $\otimes$ is the Kronecker tensor product and $\odot$ the entrywise Hadamard product. $\underline{1}_K$ is a column vector of length K with all elements equal 1, and $\mathbf{I}$ is the identity matrix. Moreover, * stands for conjugate, ^{T} for transpose and ^{H} for conjugate transpose. x_i is the i-th element of vector $\underline{x}$. $\underline{y} = \exp(\underline{x})$ and $\underline{z} = \sqrt{\underline{x}}$ are element-by-element operations, i.e. $y_i = \exp(x_i)$ and $z_i = \sqrt{x_i}$, respectively. $\mathrm{diag}(\underline{x})$ is a diagonal matrix containing the elements of $\underline{x}$.

II. Signal Model

We consider a TDM-MIMO radar which consists of a linear array with N_{Tx} transmitting (Tx) and N_{Rx} receiving (Rx) colocated, isotropic antennas. The radar transmits a narrowband signal which is reflected by a target, moving relative to the radar. The target is modeled as a point target. ϑ is the DOA of the target, measured perpendicular to the linear array. λ is the carrier wavelength. The positions of the Rx and Tx antennas are denoted by $\underline{d}^{\text{Rx}} \in \mathbb{R}^{N_{\text{Rx}}}$ and $\underline{d}^{\text{Tx}} \in \mathbb{R}^{N_{\text{Tx}}}$, respectively, in the unit of $\frac{\lambda}{2\pi}$. The Tx antennas transmit a sequence of N_{Pulse} pulses at the time instances $\underline{t} \in \mathbb{R}^{N_{\text{Pulse}}}$, with pulse durations $\underline{\Delta} \in \mathbb{R}^{N_{\text{Pulse}}}$ and power $\underline{p} \in \mathbb{R}^{N_{\text{Pulse}}}$. $\underline{d}^{\text{Pulse}} \in \mathbb{R}^{N_{\text{Pulse}}}$ are the positions of the Tx antennas in the sequence in which they transmit. If an antenna transmits more than once, its position occurs several times in $\underline{d}^{\text{Pulse}}$, see Fig. 1 for an example. Note that $\underline{d}^{\text{Pulse}}$ and $\underline{t}$ can be chosen independently. The Rx steering vector is $\underline{a}^{\text{Rx}}(u) = \exp(j\,\underline{d}^{\text{Rx}} u)$ with the electrical angle $u = \sin(\vartheta)$. The Tx steering vector with transmission sequence

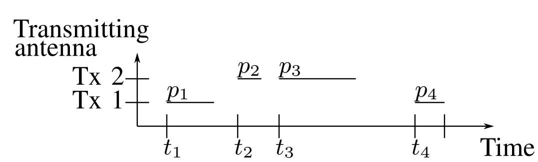

Fig. 1. Example of a TDM scheme: 2 Tx antennas transmitting at times $\underline{t} = [t_1, t_2, t_3, t_4]^T$ with power $\underline{p} = [p_1, p_2, p_3, p_4]^T$. Here, $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, d_2^{\text{Tx}}, d_2^{\text{Tx}}, d_1^{\text{Tx}}]^T$.

$\underline{d}^{\text{Pulse}}$ is $\underline{a}^{\text{Pulse}}(u) = \exp(j\,\underline{d}^{\text{Pulse}}u)$. The model contains L measurement cycles. Each cycle consists of N_{Pulse} pulses. The baseband signal of cycle l is given by

$$\underline{X}(l) = \left\{\sqrt{\underline{\rho}} \odot \exp(j\,\underline{t}\,\omega) \odot \underline{a}^{\text{Pulse}}(u)\right\} \otimes \underline{a}^{\text{Rx}}(u)\,s(l) + \underline{N}(l), \quad l = 1, \ldots, L. \tag{1}$$

Here, $s(l) \in \mathbb{C}$ is the unknown, deterministic complex target signal and $\underline{N}(l)$ the noise. $\underline{\rho} \in \mathbb{R}^{N_{\text{Pulse}}}$ contains the transmission energies of the pulses, i.e. ρ_i is the transmission energy of pulse i. It can be expressed as the product of $\underline{p}$ and $\underline{\Delta}$: $\underline{\rho} = \underline{p} \odot \underline{\Delta}$. The expression $\exp(j\,\underline{t}\,\omega)$ is the phase change due to the Doppler frequency ω. Defining the new steering vector

$$\underline{b}(u,\omega) = \left\{\sqrt{\underline{\rho}} \odot \exp(j\,\underline{t}\,\omega) \odot \underline{a}^{\text{Pulse}}(u)\right\} \otimes \underline{a}^{\text{Rx}}(u) \in \mathbb{C}^{N_{\text{Virt}}}, \tag{2}$$

$$N_{\text{Virt}} = N_{\text{Pulse}} \cdot N_{\text{Rx}}, \tag{3}$$

the baseband signal can be written as

$$\underline{X}(l) = \underline{b}(u,\omega)\,s(l) + \underline{N}(l). \tag{4}$$

For convenience we write $\underline{b}(u,\omega)$ as

$$\underline{b}(u,\omega) = \sqrt{\underline{\rho}^{\text{Virt}}} \odot \exp(j\,\underline{t}^{\text{Virt}}\,\omega) \odot \underline{a}^{\text{Virt}}(u), \tag{5}$$

where $\underline{\rho}^{\text{Virt}} = \underline{\rho} \otimes \underline{1}_{N_{\text{Rx}}}$, $\underline{t}^{\text{Virt}} = \underline{t} \otimes \underline{1}_{N_{\text{Rx}}}$ and the steering vector of the virtual array $\underline{a}^{\text{Virt}}(u) = \underline{a}^{\text{Pulse}}(u) \otimes \underline{a}^{\text{Rx}}(u)$. Note that $\underline{a}^{\text{Virt}}(u) = \exp(j\,\underline{d}^{\text{Virt}}u)$ with the virtual array

$$\underline{d}^{\text{Virt}} = \underline{1}_{N_{\text{Pulse}}} \otimes \underline{d}^{\text{Rx}} + \underline{d}^{\text{Pulse}} \otimes \underline{1}_{N_{\text{Rx}}} \in \mathbb{R}^{N_{\text{Virt}}}. \tag{6}$$

We make the following assumptions:

- $\underline{N}(l)$ is circular complex Gaussian with zero mean, spatially and temporally uncorrelated with $\mathrm{E}\left(\underline{N}(l)\ \underline{N}^H(m)\right) = \delta_{l,m}\ \sigma^2\mathbf{I}$.
- The distance of the target to the MIMO radar is much larger than the aperture of the radar. Thus, the target's DOA ϑ and also the electrical angle u are the same for all antennas. Moreover, the target's radar cross section is the same for all antennas.
- The DOA ϑ does not change significantly during the L measurement cycles, i.e. the change is much smaller than the DOA accuracy of the radar and thus it is negligible.
- The target's relative radial velocity is constant during the L measurement cycles. Thus, the Doppler frequency ω is constant.

Therefore, the unknown quantities to be estimated are

$$\underline{\Theta} = [u, \omega, s(1), \ldots, s(L), \sigma^2]^T. \tag{7}$$

III. Cramer-Rao Bound

The CRB is a lower bound for the covariance matrix of any unbiased estimator $\widehat{\underline{\Theta}}$ [6]. In our model, the unknown parameter vector is $\underline{\Theta}$ in (7). We want to obtain that part of the CRB of $\underline{\Theta}$ which corresponds to the parameters u, ω. Thus, we have to derive the Fisher Information Matrix (FIM) $\mathbf{J}$ of the whole parameter vector $\underline{\Theta}$ and compute its inverse $\mathbf{J}^{-1}$. After that, we take that 2×2 block corresponding to u, ω. This part is denoted by $\mathrm{CRB}_{u,\omega}$. Following [7], it can be computed by

$$\mathrm{CRB}_{u,\omega}^{-1} = 2L\,\frac{\sigma_s^2}{\sigma^2}\,\mathrm{Re}\,(\mathbf{C}) \tag{8}$$

where

$$\mathbf{C} = \mathbf{D}^H\mathbf{P}_{\underline{b}}^{\perp}\mathbf{D}, \tag{9}$$

$$\mathbf{D} = \left[\frac{\partial \underline{b}(u,\omega)}{\partial u}, \quad \frac{\partial \underline{b}(u,\omega)}{\partial \omega}\right], \tag{10}$$

$$\mathbf{P}_{\underline{b}}^{\perp} = \mathbf{I} - \underline{b}\,(\underline{b}^H\ \underline{b})^{-1}\,\underline{b}^H, \tag{11}$$

$$\sigma_s^2 = \frac{1}{L}\sum_{l=1}^{L} |s(l)|^2. \tag{12}$$

Computations show that $\mathbf{C}$ is real, hence

$$\mathrm{CRB}_{u,\omega}^{-1} = 2L\,\frac{\sigma_s^2}{\sigma^2}\,\mathbf{C} \tag{13}$$

and

$$\mathbf{C} = N_{\text{Rx}}(\underline{1}^T\underline{\rho})\left\{\begin{bmatrix}\mathrm{Var}^{\mathrm{S}}(\underline{d}^{\text{Rx}}) & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}\mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\text{Pulse}},\underline{\rho}) & \mathrm{Cov}^{\mathrm{WS}}(\underline{t},\underline{d}^{\text{Pulse}},\underline{\rho})\\ \mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\text{Pulse}},\underline{t},\underline{\rho}) & \mathrm{Var}^{\mathrm{WS}}(\underline{t},\underline{\rho})\end{bmatrix}\right\}. \tag{14}$$

Here we use the following definitions for some vectors $\underline{x}, \underline{y}, \underline{w} \in \mathbb{C}^K$:

- weighted sample mean
$$\mathrm{E}^{\mathrm{WS}}(\underline{x},\underline{w}) := \frac{1}{\underline{1}^T\underline{w}}\underline{w}^T\underline{x}, \tag{15}$$
- weighted sample correlation
$$\mathrm{Corr}^{\mathrm{WS}}(\underline{x},\underline{y},\underline{w}) := \frac{1}{\underline{1}^T\underline{w}}\ \underline{y}^H\,\mathrm{diag}(\underline{w})\underline{x}, \tag{16}$$
- weighted sample covariance
$$\mathrm{Cov}^{\mathrm{WS}}(\underline{x},\underline{y},\underline{w}) := \mathrm{Corr}^{\mathrm{WS}}\left(\underline{x} - \underline{1}\,\mathrm{E}^{\mathrm{WS}}(\underline{x},\underline{w}), \underline{y} - \underline{1}\,\mathrm{E}^{\mathrm{WS}}(\underline{y},\underline{w}), \underline{w}\right), \tag{17}$$
- and weighted sample variance
$$\mathrm{Var}^{\mathrm{WS}}(\underline{x},\underline{w}) := \mathrm{Cov}^{\mathrm{WS}}(\underline{x},\underline{x},\underline{w}). \tag{18}$$

If $\underline{w} \propto \underline{1}_K$, all weights are equal and the weighted sample mean, covariance and variance turn into simple sample mean $\mathrm{E}^{\mathrm{S}}(\underline{x})$, sample covariance $\mathrm{Cov}^{\mathrm{S}}(\underline{x},\underline{y})$ and sample variance $\mathrm{Var}^{\mathrm{S}}(\underline{x})$, respectively. We are interested in the CRB of the electrical angle u. Assuming $\det \mathrm{CRB}_{u,\omega}^{-1} \neq 0$, from (13) and (14)

$$\mathrm{CRB}_u = [\mathrm{CRB}_{u,\omega}]_{11} = \frac{1}{2L}\frac{1}{S}\frac{1}{U} \tag{19}$$

follows, with

$$S = \frac{\sigma_s^2}{\sigma^2}N_{\text{Rx}}\left(\underline{1}^T\underline{\rho}\right), \tag{20}$$

$$U = \mathrm{Var}^{\mathrm{S}}(\underline{d}^{\text{Rx}}) + \mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\text{Pulse}},\underline{\rho}) - \frac{\left(\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\text{Pulse}},\underline{t},\underline{\rho})\right)^2}{\mathrm{Var}^{\mathrm{WS}}(\underline{t},\underline{\rho})}. \tag{21}$$

S is the total SNR and U depends on the positions of the Rx and Tx antennas as well as on the weighted covariance between $\underline{d}^{\text{Pulse}}$ and $\underline{t}$.

IV. Optimal TDM Schemes

The TDM scheme is characterized by the parameters

$$\underline{\vartheta}^{\text{TDM}} = [\underline{d}^{\text{Pulse}\,T}, \underline{t}^T, \underline{\rho}^T, N_{\text{Pulse}}]^T \tag{22}$$

with $\underline{d}^{\text{Pulse}}, \underline{t}, \underline{\rho} \in \mathbb{R}^{N_{\text{Pulse}}}$. Note that $\underline{\vartheta}^{\text{TDM}}$ contains both discrete variables (N_{Pulse}) and continuous variables ($\underline{t}, \underline{\rho}$, positions in $\underline{d}^{\text{Pulse}}$). First we compare the TDM-MIMO radar to different radar systems. Moreover, we are interested in TDM schemes which minimize CRB_u. In the following, we call such a TDM scheme an optimal TDM scheme. Typically there are constraints on some of the parameters $\underline{\vartheta}^{\text{TDM}}$. In general, different constraints result in a different minimal value of CRB_u. We derive conditions for optimal TDM schemes for two different cases.

A. Comparison of CRB_u to the CRB of a SIMO Radar and a MIMO Radar with a Stationary Target

We compare the TDM-MIMO radar to a MIMO radar with a stationary target and to a SIMO radar. For that purpose, we compute the appropriate CRBs.

1) MIMO Radar, Stationary Target: If the target is stationary, i. e. it does not move relative to the radar, the Doppler frequency vanishes, i. e. $\omega = 0$. If this is known a priori, there is no need to estimate it. In that case we can use the model (4), setting $\omega = 0$. Then the CRB of the electrical angle $\text{CRB}_{u,\text{stat}}$ can be computed analogously to the moving target case. This results in

$$\text{CRB}_{u,\text{stat}} = \frac{1}{2L}\frac{1}{S}\frac{1}{U_{\text{stat}}} \qquad \in \mathbb{R}, \tag{23}$$

$$S = \frac{\sigma_s^2}{\sigma^2} N_{\text{Rx}} \left(\underline{1}^T \underline{\rho}\right) \in \mathbb{R}, \tag{24}$$

$$U_{\text{stat}} = \text{Var}^{\text{S}}(\underline{d}^{\text{Rx}}) + \text{Var}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{\rho}) \in \mathbb{R}. \tag{25}$$

2) SIMO Radar: The CRB for a SIMO radar can be deduced from the CRB for the MIMO radar by using only one Tx antenna, i. e. setting $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, \dots, d_1^{\text{Tx}}]^T$. By doing so, we do not include the beamforming gain of a phased array with N_{Tx} Tx antennas. This would be only possible if the target's DOA was known approximately a priori, which we do not assume here. In this case, $\text{Cov}^{\text{WS}}(\underline{t}, \underline{d}^{\text{Pulse}}, \underline{\rho}) = \underline{0}$, i. e. the DOA and the Doppler frequency decouple always. This is logical since there is no information gain on the DOA by different Tx positions. Thus, the movement of the target does not influence the CRB of the electrical angle. The part of the CRB corresponding to the electrical angle is given by

$$\text{CRB}_{u,\text{SIMO}} = \frac{1}{2\,L}\frac{1}{S}\frac{1}{U_{\text{SIMO}}} \quad \in \mathbb{R}, \tag{26}$$

$$S = \frac{\sigma_s^2}{\sigma^2} N_{\text{Rx}} \left(\underline{1}^T \underline{\rho}\right) \in \mathbb{R}, \tag{27}$$

$$U_{\text{SIMO}} = \text{Var}^{\text{S}}(\underline{d}^{\text{Rx}}) \qquad \in \mathbb{R}, \tag{28}$$

regardless of the movement of the target.

3) Comparison: We compare the different CRBs CRB_u, $\text{CRB}_{u,\text{stat}}$ and $\text{CRB}_{u,\text{SIMO}}$. They differ only in the term U, U_{stat} and U_{SIMO}, respectively. In the SIMO radar, only the Rx antenna positions $\underline{d}^{\text{Rx}}$ are relevant. In the stationary MIMO case, there is an additional term $\text{Var}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{\rho})$ due to the different Tx positions. The MIMO radar with the non-stationary target has the same term as well but also an additional coupling term U_{penalty}:

$$U = U_{\text{stat}} - U_{\text{penalty}}, \tag{29}$$

$$U_{\text{penalty}} = \frac{\left(\text{Cov}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{t}, \underline{\rho})\right)^2}{\text{Var}^{\text{WS}}(\underline{t}, \underline{\rho})}. \tag{30}$$

This leads in general to a degradation. If the coupling vanishes, i. e. $\text{Cov}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{t}, \underline{\rho}) = 0$, the CRBs for the non-stationary and stationary target are equal, $\text{CRB}_u = \text{CRB}_{u,\text{stat}}$.

Theorem 1. *For a fixed $\underline{\vartheta}^{TDM}$ the following inequalities hold for CRB_u*

$$CRB_{u,SIMO} \geq CRB_u \geq CRB_{u,stat}. \tag{31}$$

Moreover,

$$CRB_u = CRB_{u,stat} \text{ iff } \text{Cov}^{\text{WS}}(\underline{d}^{Pulse}, \underline{t}, \underline{\rho}) = 0. \tag{32}$$

The proof is given in the appendix. The theorem states that for a given TDM scheme, the CRB for a moving target is always bounded by the CRB of the corresponding SIMO radar and the CRB of the same MIMO radar with the same TDM scheme, but with a stationary target. Hence we can easily compute the MIMO gain compared to the SIMO radar and the loss in accuracy due to the movement of the target. By using the same MIMO radar hardware (N_{Tx} Tx antennas and N_{Rx} Rx antennas) the choice of the TDM scheme $\underline{\vartheta}^{\text{TDM}}$ has a significant impact on the DOA accuracy. In the worst case $\text{CRB}_u = \text{CRB}_{u,\text{SIMO}}$ and in the best case $\text{CRB}_u = \text{CRB}_{u,\text{stat}}$.

4) Examples: We present 2 examples of the TDM scheme which actually reach the bounds of Theorem 1. We consider an uniform linear array (ULA) of 4 Tx antennas with $N_{\text{Pulse}} = 4$, $\underline{\rho} \propto \underline{1}$ and $\underline{t} \propto [0, 1, 2, 3]^T$.

Bad TDM Schemes: We set $\underline{d}^{\text{Pulse}} = c_1\,\underline{t} + c_2$ with constants c_1, c_2. Then, $U_{\text{penalty}} = \text{Var}^{\text{S}}(\underline{d}^{\text{Pulse}})$ and $\text{CRB}_u = \text{CRB}_{u,\text{SIMO}}$. This is the case if the Tx antennas transmit at equally spaced time instants in the order of their geometric arrangement .

Good TDM Scheme: The lower bound in (31) is reached when using the TDM scheme $\underline{d}^{\text{Pulse}} = [d_1^{\text{Tx}}, d_4^{\text{Tx}}, d_4^{\text{Tx}}, d_1^{\text{Tx}}]^T$.

Both TDM schemes are depicted in Fig. 2.

B. Optimal TDM Schemes Under Limited Transmission Energy and Aperture Size

To achieve a high DOA accuracy, CRB_u has to be minimized. Theorem 1 states a condition under which $\text{CRB}_u = \text{CRB}_{u,\text{stat}}$. $\underline{\vartheta}^{\text{TDM}}$ contains further degrees of freedom which can be optimized to minimize CRB_u. We determine the minimum value of CRB_u w. r. t. $\underline{\vartheta}^{\text{TDM}}$. Moreover, we derive conditions

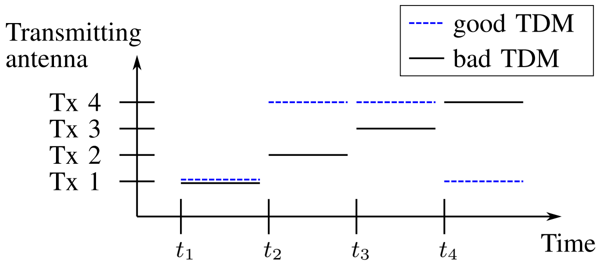

Fig. 2. Good and bad TDM scheme for an ULA of 4 Tx antennas. $\underline{t} \propto [0,1,2,3]^T$.

on $\underline{\vartheta}^{\text{TDM}}$ under which this minimum is reached. We consider TDM schemes which satisfy the energy constraint

$$\underline{1}^T \underline{\rho} \leq \rho^{\max}, \tag{33}$$

i. e. the transmission energy per cycle is limited by $\rho^{\max}$. Note that CRB_u in (19) is scaled by $1/\left(\underline{1}^T \underline{\rho}\right)$. Obviously this is a physically meaningful constraint. Moreover, the Tx antennas have to be placed within the real aperture range $\left[d^{\text{Tx}}_{\text{min}}, d^{\text{Tx}}_{\text{max}}\right]$, i. e.

$$d^{\text{Tx}}_{\text{min}} \leq d^{\text{Pulse}}_i \leq d^{\text{Tx}}_{\text{max}}, \quad i = 1, \ldots, N_{\text{Pulse}}. \tag{34}$$

Theorem 2.

1) *CRB_u is minimal w. r. t. $\underline{\vartheta}^{TDM}$ if and only if*
   a) *only the 2 outermost Tx antennas with $d^{Tx}_1 = d^{Tx}_{min}$ and $d^{Tx}_2 = d^{Tx}_{max}$ are used,*
   b) $\underline{1}^T \underline{\rho}^{(1)} = \underline{1}^T \underline{\rho}^{(2)}$,
   c) $\underline{1}^T \underline{\rho} = \rho^{max}$,
   d) $\text{E}^{\text{WS}}(\underline{t}^{(1)}, \underline{\rho}^{(1)}) = \text{E}^{\text{WS}}(\underline{t}^{(2)}, \underline{\rho}^{(2)})$,

   where $\underline{t}^{(1)}$ and $\underline{t}^{(2)}$ are the time instances when the 1. and 2. Tx antenna transmits, respectively. $\underline{\rho}^{(1)}$ and $\underline{\rho}^{(2)}$ are the energies of the transmitted pulses of the 1. and 2. Tx antenna, respectively.
2) *The minimal value of CRB_u w. r. t. $\underline{\vartheta}^{TDM}$ is given by*

$$CRB_{u,min} = \frac{1}{2L} \frac{1}{S_{max}} \frac{1}{U_{max}}, \tag{35}$$

$$S_{max} = \frac{\sigma_s^2}{\sigma^2} N_{Rx} \rho^{max}, \tag{36}$$

$$U_{max} = \text{Var}^{\text{S}}(\underline{d}^{Rx}) + \frac{1}{4}\left(d^{Tx}_{max} - d^{Tx}_{min}\right)^2. \tag{37}$$

The proof is presented in the appendix. As an example for the definition of $\underline{t}^{(1)}$ and $\underline{t}^{(2)}$ consider the TDM scheme depicted in Fig. 1: There, $\underline{t}^{(1)} = [t_1, t_4]^T$ and $\underline{t}^{(2)} = [t_2, t_3]^T$. The conditions of the theorem can be interpreted in the following way: Using the two outermost Tx antennas makes the virtual aperture as large as possible. $\underline{1}^T \underline{\rho}^{(1)} = \underline{1}^T \underline{\rho}^{(2)}$ means that the same amount of energy is transmitted from each Tx antenna. Roughly spoken, this yields the same amount of spatial information for both Tx antennas. $\underline{1}^T \underline{\rho} = \rho^{\max}$ states to transmit as much energy as possible, which obviously leads to the highest possible SNR. The condition $\text{E}^{\text{WS}}(\underline{t}^{(1)}, \underline{\rho}^{(1)}) = \text{E}^{\text{WS}}(\underline{t}^{(2)}, \underline{\rho}^{(2)})$ means that the weighted mean transmission time of the 1. and 2. antenna are the same. Roughly, $\text{E}^{\text{WS}}(\underline{t}^{(1)}, \underline{\rho}^{(1)})$ is the effective measurement time instance of the data gained from the 1. Tx antenna. Hence the condition means that the effective measurement time instance is the same for both antennas. In average this results in the same Doppler phase for both Tx antennas and therefore the movement of the target does not influence the angle accuracy. From this condition follows immediately $\text{Cov}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{t}, \underline{\rho}) = 0$ and thus $\text{CRB}_u = \text{CRB}_{u,\text{stat}}$, where $\text{CRB}_{u,\text{stat}}$ is the CRB of the electrical angle of a stationary target using the same radar and TDM scheme. Thus, the achievable DOA accuracy is as large as if the target is not moving relative to the radar and the complete virtual aperture of the MIMO radar can be used as indicated in (37).

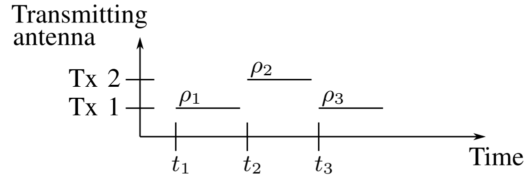

Fig. 3. Example of a TDM scheme satisfying the conditions of Theorem 2 with $\underline{t} \propto [0,1,2]^T$ and $\underline{\rho} = \frac{\rho^{\max}}{4}[1,2,1]^T$.

Theorem 2 helps us to find TDM schemes which achieve the smallest possible CRB_u. If we optimize $\underline{\vartheta}^{\text{TDM}}$, this theorem tells us which conditions have to be satisfied in order to yield an optimal TDM scheme.

We give two examples of optimal TDM schemes.

Example 1: The TDM scheme uses $N_{\text{Pulse}} = 4$ pulses and the same energy per pulse, $\underline{\rho} = \frac{\rho^{\max}}{4} \underline{1}$. Moreover, $\underline{d}^{\text{Pulse}} = [d^{\text{Tx}}_{\text{min}}, d^{\text{Tx}}_{\text{max}}, d^{\text{Tx}}_{\text{max}}, d^{\text{Tx}}_{\text{min}}]^T$, $\underline{t} \propto [0,1,2,3]^T$. The transmission sequence is already depicted as the good TDM scheme in Fig. 2 with $d^{\text{Tx}}_1 = d^{\text{Tx}}_{\text{min}}$, $d^{\text{Tx}}_4 = d^{\text{Tx}}_{\text{max}}$. Since the energy per pulse is constant, we can easily see that condition 1d is satisfied due to the symmetry of the good TDM scheme in Fig. 2.

Example 2: The radar transmits $N_{\text{Pulse}} = 3$ pulses with different pulse energies, $\underline{\rho} = \frac{\rho^{\max}}{4}[1,2,1]^T$, $\underline{t} \propto [0,1,2]^T$ and $\underline{d}^{\text{Pulse}} = [d^{\text{Tx}}_{\text{min}}, d^{\text{Tx}}_{\text{max}}, d^{\text{Tx}}_{\text{min}}]^T$. This is depicted in Fig. 3.

C. Optimal TDM Schemes With Constant Transmission Energy and Equidistant Transmission Times

Theorem 2 states conditions for optimal TDM schemes under two physically meaningful constraints. Typical applications introduce additional constraints on the TDM scheme. In general, the additional constraints result in other optimal TDM schemes. In the following, we search for optimal TDM schemes under the constraints (33), (34) and additional practical constraints: We consider TDM schemes which contain pulses transmitted at equidistant time instances with the same transmission energy per pulse. This is e. g. the case in automobile radar systems using multi-chirp sequences, i.e. several subsequent frequency modulated continuous wave pulses. We consider two cases: a) the number of pulses N_{Pulse} in the TDM scheme is given and b) can be chosen freely.

Since $\underline{\rho} \propto \underline{1}$, the functions E^{WS}, Var^{WS} and Cov^{WS} simplify to E^{S}, Var^{S} and Cov^{S}, respectively. Without loss of generality we set $\underline{t} = [1, \ldots, N_{\text{Pulse}}]^T - \text{E}^{\text{S}}([1, \ldots, N_{\text{Pulse}}]^T)\underline{1}$ and $d^{\text{Tx}}_{\text{min}} = -1$ and $d^{\text{Tx}}_{\text{max}} = 1$. Analog to Theorem 2, the optimum is achieved by using only the two outermost Tx antennas such that $\underline{d}^{\text{Pulse,opt}} \in \{-1, 1\}^{N_{\text{Pulse}}}$. This simplifies significantly the optimization problem, since the number of Tx antennas and their positions are already fixed. For a given number of pulses N_{Pulse}, we derive the conditions TDM schemes have to fulfill in order to minimize CRB_u. Dependent on N_{Pulse} the TDM schemes achieve different values of CRB_u. We show for which values of N_{Pulse} the smallest CRB_u is achieved, which is case b).

To minimize CRB_u in (19), S in (20) and U in (21) have to be maximized. S is maximized iff $\underline{\rho} = \frac{\rho^{\max}}{N_{\text{Pulse}}}\underline{1}$. To maximize U, we have to maximize

$$\Omega = \text{Var}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{\rho}) - \frac{\left(\text{Cov}^{\text{WS}}(\underline{d}^{\text{Pulse}}, \underline{t}, \underline{\rho})\right)^2}{\text{Var}^{\text{WS}}(\underline{t}, \underline{\rho})}. \tag{38}$$

For different N_{Pulse}, the maximal value of Ω and the criteria for an optimal TDM scheme differ. Therefore, we have to consider different cases of N_{Pulse}.

Theorem 3.
$N_{\textit{Pulse}} = 4n, n \in \mathbb{N}$ *An optimal TDM scheme has to use both Tx elements equally often (balanced),* $\text{Var}^{\text{S}}(\underline{d}^{\textit{Pulse}}) = 1$*, and fulfill* $\text{Cov}^{\text{S}}(\underline{d}^{\textit{Pulse}}, \underline{t}) = 0$ *which is equivalent to* $\text{E}^{\text{S}}(\underline{t}^{(1)}) = \text{E}^{\text{S}}(\underline{t}^{(2)})$. *The optimal value for* Ω *is*

$$\Omega(\underline{d}^{\textit{Pulse,opt}})\Big|_{N_{\textit{Pulse}} \in 4\mathbb{N}} = 1. \tag{39}$$

$N_{\textit{Pulse}} = 4n - 2, n \in \mathbb{N}$ *An optimal TDM scheme has to be balanced and fulfill* $\text{Cov}^{\text{S}}(\underline{d}^{\textit{Pulse}}, \underline{t}) = \pm\frac{1}{N_{\textit{Pulse}}}$. *The maximum* Ω *for the minimal* $\textit{CRB}_u$ *is*

$$\Omega(\underline{d}^{\textit{Pulse,opt}})\Big|_{N_{\textit{Pulse}} \in (4\mathbb{N}-2)} = 1 - \frac{1}{N_{\textit{Pulse}}^2} \cdot \frac{12}{N_{\textit{Pulse}}^2 - 1}. \tag{40}$$

$N_{\textit{Pulse}} = 2n + 1, n \in \mathbb{N}$ *In an optimal TDM scheme, one Tx antenna has to transmit one pulse more than the other Tx antenna and the TDM scheme has to fulfill* $\text{E}^{\text{S}}(\underline{t}^{(1)}) = \text{E}^{\text{S}}(\underline{t}^{(2)})$. *The maximum* Ω *for the minimal* $\textit{CRB}_u$ *is*

$$\Omega(\underline{d}^{\textit{Pulse,opt}})\Big|_{N_{\textit{Pulse}} \in (2\mathbb{N}+1)} = 1 - \frac{1}{N_{\textit{Pulse}}^2}. \tag{41}$$

Due to lack of space, we omit the proof here. A justification for the case differentiation is presented in the appendix.

If N_{Pulse} can be chosen freely in case b), then an optimal TDM scheme has to use $N_{\text{Pulse}} = 4n$, $n \in \mathbb{N}$ pulses, since due to Theorem 3, it achieves the smallest CRB_u. We present several examples which satisfy the conditions of Theorem 3 for the different cases.

$N_{\text{Pulse}} = 4n, n \in \mathbb{N}$ Such optimal TDM schemes can be constructed by building schemes which are balanced and symmetric to the center index, i.e. $d_i^{\text{Pulse}} = d^{\text{Pulse}}_{N_{\text{Pulse}}+1-i}$. An example is $\underline{d}_1^{\text{Pulse, opt}} = [1, 1, -1, -1, -1, -1, 1, 1]^T$. $\underline{d}_2^{\text{Pulse, opt}} = [1, -1, -1, 1, 1, -1, 1, 1, -1]^T$ is also optimal, but not of this structure. Both examples are depicted in Fig. 4.

$N_{\text{Pulse}} = 4n - 2, n \in \mathbb{N}$ Such optimal TDM schemes can be constructed by building schemes which are balanced and symmetric to the center index, except for the elements which are closest to the center of the sequence. An example of such an optimal TDM scheme is $\underline{d}_1^{\text{Pulse, opt}} = [-1, 1, 1, -1, 1, -1]^T$. $\underline{d}_2^{\text{Pulse, opt}} = [-1, 1, 1, -1, -1, 1]^T$ is also optimal, but not of this structure.

$N_{\text{Pulse}} = 2n + 1, n \in \mathbb{N}$ Such optimal TDM schemes can be constructed by building schemes where one antenna transmits one pulse more than the other and which are symmetric to the center index. An example is $\underline{d}_1^{\text{Pulse, opt}} = [-1, -1, 1, 1, 1, -1, -1]^T$. $\underline{d}_2^{\text{Pulse, opt}} = [-1, 1, 1, -1, -1, -1, 1]^T$ is also optimal, but not of this structure.

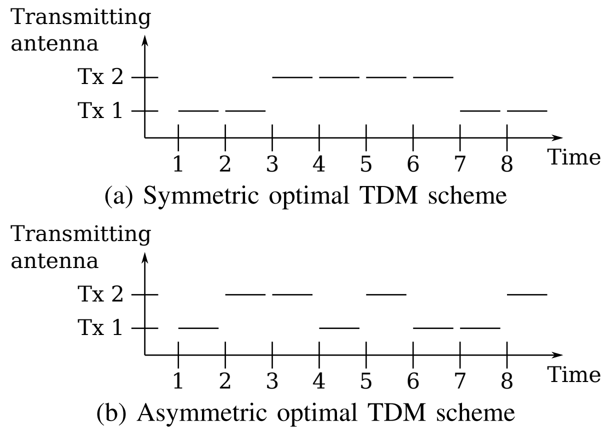

Fig. 4. Example of optimal TDM schemes with $N_{\text{Pulse}} = 8$

V. Simulations

To verify and foster the understanding of the theoretical results, we present some numerical simulations. The MIMO radar under consideration consists of 6 Rx and 6 Tx antennas, which are uniformly spaced with an antenna distance of $\lambda/2$, i.e. $\underline{d}^{\text{Rx}} = \underline{d}^{\text{Tx}} = \pi \cdot [0, 1, 2, 3, 4, 5]^T$. We choose $L = 1$ cycle and consider TDM schemes with $N_{\text{Pulse}} = 6$ pulses and $\underline{\rho} \propto \underline{1}$. The transmission time instants are $\underline{t} = [0, 1, \ldots, N_{\text{Pulse}} - 1]^T$, where we have normalized $\underline{t}$ such that it is dimensionless. Note that these are the constraints considered in Theorem 3. We take the bad TDM scheme with $\underline{d}^{\text{Pulse}} = \underline{d}^{\text{Tx}}$, similar to the bad TDM scheme depicted in Fig. 2 and the optimal TDM scheme $\underline{d}^{\text{Pulse}} = [d^{\text{Tx}}_{\text{min}}, d^{\text{Tx}}_{\text{max}}, d^{\text{Tx}}_{\text{max}}, d^{\text{Tx}}_{\text{min}}, d^{\text{Tx}}_{\text{max}}, d^{\text{Tx}}_{\text{min}}]^T$. The optimal TDM scheme is one of the examples given above for Theorem 3. The electrical angle of the target is $u = \sin(10°)$ and the Doppler frequency is $\omega = 1.3$. For each TDM scheme, we compute the root mean square error (RMSE) of the deterministic Maximum Likelihood (ML) estimate of u by doing Monte-Carlo simulations. Note that the unknown parameter vector of the ML estimator is $\underline{\Theta}$ in (7) and not u alone. We do this for different values of the total SNR S and compare the RMSE with CRB_u. The result is depicted in Fig. 5. The plot shows that the ML estimators achieve the corresponding CRBs above a threshold of approximately 16 and 18 dB, respectively. Moreover, the optimal TDM scheme

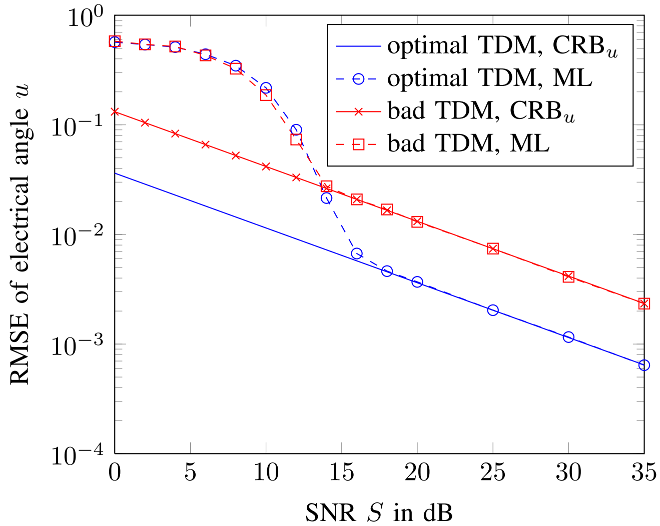

Fig. 5. RMSE of ML and CRB of a bad and optimal TDM scheme

achieves a higher accuracy than the bad one. In this example, this corresponds to an SNR gain of approximately 11.2 dB. Hence by choosing a good TDM sequence, the DOA accuracy can be significantly increased without changing the Rx or Tx array.

VI. Conclusion and Outlook

We have investigated the DOA estimation of a moving target using a TDM-MIMO radar. We analyzed the achievable DOA accuracy by computing the CRB. We have compared it to the CRB of a SIMO radar and a MIMO radar with a stationary target. Under different constraints, we have derived conditions for optimal TDM schemes such that the lowest possible value of the CRB is achieved. Numerical simulations confirmed the analytical results.

Appendix

Proof of Theorem 1

For arbitrary but fixed $\underline{\rho}$ we can assume w. l. o. g. $\mathrm{E}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{\rho}) = \mathrm{E}^{\mathrm{WS}}(\underline{t}, \underline{\rho}) = 0$. Then the relevant weighted covariance $\mathrm{Cov}^{\mathrm{WS}}$ and variance $\mathrm{Var}^{\mathrm{WS}}$ terms simplify to weighted correlations $\mathrm{Corr}^{\mathrm{WS}}$. For fixed $\underline{\rho}$, $\mathrm{Corr}^{\mathrm{WS}}(\underline{x}, \underline{y}, \underline{\rho})$ is an inner product. Hence we can use the Cauchy Schwarz inequality, from which the first part of the Theorem follows. The second part follows immediately from (29).

Proof of Theorem 2

1. To minimize CRB_u in (19), S in (20) and U in (21) have to be maximized. S is maximal iff $\underline{1}^T \underline{\rho} = \rho^{\max}$, since the other variables are fixed. This is condition 1c. U is independent of a scaling of $\underline{\rho}$. To maximize U, we have to maximize

$$\Omega = \mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{\rho}) - \frac{\left(\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho})\right)^2}{\mathrm{Var}^{\mathrm{WS}}(\underline{t}, \underline{\rho})} \qquad (42)$$

since $\underline{d}^{\mathrm{Rx}}$ is given. It can be shown that $\mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{\rho})$ is maximal iff only the two outermost Tx antennas are used and $\underline{1}^T \underline{\rho}^{(1)} = \underline{1}^T \underline{\rho}^{(2)}$. Due to lack of space we do not prove that here. Using only two Tx antennas, $\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0$ iff $\mathrm{E}^{\mathrm{WS}}(\underline{t}^{(1)}, \underline{\rho}^{(1)}) = \mathrm{E}^{\mathrm{WS}}(\underline{t}^{(2)}, \underline{\rho}^{(2)})$. The two conditions $\underline{1}^T \underline{\rho}^{(1)} = \underline{1}^T \underline{\rho}^{(2)}$ and $\mathrm{E}^{\mathrm{WS}}(\underline{t}^{(1)}, \underline{\rho}^{(1)}) = \mathrm{E}^{\mathrm{WS}}(\underline{t}^{(2)}, \underline{\rho}^{(2)})$ do not contradict. An example which fulfills both conditions is presented in Sec. IV-B. Thus, $\max_{\underline{d}^{\mathrm{Pulse}}, \underline{\rho}} \mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{\rho})$ stays the same independent whether $\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0$ is fulfilled or not. Hence $\mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{\rho})$ can be maximized and $\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0$ at the same time. Thus, Ω is maximal iff $\mathrm{Var}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{\rho})$ is maximal and at the same time $\mathrm{Cov}^{\mathrm{WS}}(\underline{d}^{\mathrm{Pulse}}, \underline{t}, \underline{\rho}) = 0$. Therefore, Ω is maximal iff conditions 1a, 1b and 1d are fulfilled.

2. The minimal value of CRB_u can be determined by choosing $\underline{\vartheta}^{TDM}$ such that the conditions of the 1. part of the Theorem are fulfilled.

Case Differentiation in Theorem 3

In the following we present the reason for the case differentiation. To maximize Ω, we have to maximize $\mathrm{Var}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}})$ and to minimize $|\,\mathrm{Cov}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}}, \underline{t})|$ at the same time. First we consider $\mathrm{Var}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}})$. For even N_{Pulse} both Tx antennas can be used equally often such that $\mathrm{Var}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}})$ is maximal, $\mathrm{Var}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}}) = 1$. For odd N_{Pulse} one antenna is used more often than the other and a sample variance of 1 cannot be achieved. Hence optimal TDM schemes have to be determined for even and odd N_{Pulse} separately. A further case differentiation is required to minimize $\mathrm{Cov}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}}, \underline{t})$. For odd N_{Pulse} it can be shown that $\min|\,\mathrm{Cov}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}}, \underline{t})| = 0$. For even N_{Pulse} with $N_{\mathrm{Pulse}} \in 4\mathbb{N}$, the minimum value is also $\min|\,\mathrm{Cov}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}}, \underline{t})| = 0$. If N_{Pulse} is even with $N_{\mathrm{Pulse}} \in 4\mathbb{N} - 2$ then $\min|\,\mathrm{Cov}^{\mathrm{S}}(\underline{d}^{\mathrm{Pulse}}, \underline{t})| = 0.5$. Thus we have to make a second case differentiation for even N_{Pulse}. Therefore, we have to consider the cases $N_{\mathrm{Pulse}} \in 4\mathbb{N}$, $N_{\mathrm{Pulse}} \in 4\mathbb{N} - 2$ and $N_{\mathrm{Pulse}} \in 2\mathbb{N} + 1$.

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