

DIFFERENT SENSOR PLACEMENT STRATEGIES FOR TDOA BASED LOCALIZATION

Bin Yang

Chair of System Theory and Signal Processing, University of Stuttgart, Germany

ABSTRACT

This paper studies different optimization strategies for the sensor placement in source localization by using time differences of arrival. It continues the works in [1, 2] and gives answers to some open questions there. In particular, we discuss the relationship between maximum Fisher information matrix, minimum Cramer-Rao bound, spherical codes, uniform angular arrays, and Platonic solids as well as their roles in optimizing the sensor placement. Various new optimum sensor array geometries are given.

Index Terms— Position measurement, delay estimation, computational geometry

1. INTRODUCTION

Methods for navigation and localization are often based on measuring the time delay (TD) of a signal caused by its travel from a transmitter (TX) to a receiver (RX) or the time difference of arrival (TDOA), i.e. the difference of a pair of time delays. In *navigation*, the transmitters have known positions and the receiver has to be located. In *localization*, signals travel from a single transmitter, whose location is to be estimated, to several receivers at known positions. Table 1 summarizes these four cases and lists some applications. Since the distinction between transmitter and receiver is irrelevant in this paper, we treat navigation and localization in the same way and speak simply of *source localization*. The term *source* refers to the object to be located and the other objects at known positions are called *sensors*.

	TD	TDOA
Navigation >1 TXs 1 RX	GPS, mobile positioning	LORAN, DECCA, mobile positioning
Localization 1 TX >1 RXs	active radar, sonar, and seismic	passive radar, sonar, and seismic, microphone array

Table 1. Navigation and localization

In this paper, we focus on TDOA based localization. The overall localization accuracy depends on three factors: a) the accuracy of the TDOA estimates; b) the choice of the location estimator; c) the sensor positioning relative to the source.

While much research effort has been spent on developing suitable TDOA and location estimators in the last decades (see the references in [1, 2]), little was known about the impact of the sensor positioning. Most results known in the literature were obtained either numerically in terms of error contours for some simple sensor arrays [3, 4, 5] or for a fixed sensor array geometry like hexagonal cells in wireless mobile positioning [6].

In [1, 2], the impact of the sensor placement to the localization accuracy has been analysed theoretically. Conditions for optimum arrays minimizing the Cramer-Rao bound (CRB) have been derived. Examples of such optimum arrays are given. Unfortunately, several questions remain open, in particular for three-dimensional sensor arrays:

- No optimum arrays could be found for an odd number of sensors when using the full TDOA set.
- No optimum arrays are known when using the spherical TDOA set.
- Are there other good strategies for the sensor placement than the minimization of the CRB?
- What is the relationship between these strategies?

This paper gives answers to the above questions.

2. TDOA BASED LOCALIZATION

M sensors at the positions $\underline{q}_i \in \mathbb{R}^D$ ($i = 1, \dots, M$) measure signals transmitted by a source located at $\underline{p} \in \mathbb{R}^D$. $D = 2$ or 3 is the space dimension. If the TDOA measurement errors are Gaussian and uncorrelated with equal variance σ^2 , the CRB for the source location vector \underline{p} is [1]

$$\mathbf{J}^{-1} = (v\sigma)^2 (\mathbf{G}\mathbf{G}^T)^{-1} \quad (1)$$

with

$$\mathbf{G} = [\underline{g}_{ij} \ \dots]_{(i,j) \in \mathcal{I}}, \quad \underline{g}_{ij} = \underline{g}_i - \underline{g}_j, \quad \underline{g}_i = \frac{\underline{q}_i - \underline{p}}{\|\underline{q}_i - \underline{p}\|}. \quad (2)$$

\mathbf{J} is the Fisher information matrix (FIM). \underline{g}_i is a unit-length column vector pointing from the source to sensor i . \underline{g}_{ij} is the difference between two such *direction vectors*. The set \mathcal{I} contains the indexes of these sensor pairs (i, j) whose TDOA estimates contribute to the source localization. The matrix \mathbf{G} contains all vectors \underline{g}_{ij} with $(i, j) \in \mathcal{I}$. Depending on the location estimator used, the *full TDOA set*

$$\mathcal{I}_0 = \{(i, j) | 1 \leq j < i \leq M\} \quad (3)$$

containing all $M(M - 1)/2$ sensor pairs or the *spherical TDOA set*

$$\mathcal{I}_s = \{(1, k), \dots, (k - 1, k), (k + 1, k), \dots, (M, k)\} \quad (4)$$

involving $M - 1$ sensor pairs with the common *reference sensor* k are frequently used in practice.

3. SENSOR PLACEMENT STRATEGIES FOR \mathcal{I}_0

Different strategies can be considered to optimize the localization accuracy. The classical approach minimizes the trace of the CRB:

$$\min_{\{\underline{g}_i\}} f_{\text{CRB}} = \text{tr}[\mathbf{J}^{-1}] = (v\sigma)^2 \text{tr}[(\mathbf{G}\mathbf{G}^T)^{-1}]. \quad (5)$$

Alternatively, we can also maximize the trace of the FIM:

$$\max_{\{\underline{g}_i\}} f_{\text{FIM}} = \text{tr}[\mathbf{J}] = \frac{1}{(v\sigma)^2} \text{tr}[\mathbf{G}\mathbf{G}^T]. \quad (6)$$

Is this a meaningful measure?

Since all sensor pairs are used in localization due to $\mathcal{I} = \mathcal{I}_0$ and all TDOA measurements are assumed to have equal variance, all sensors are equally important for localization. Intuitively, the sensors should be placed some how uniformly on the unit circle or unit sphere. In the 2D case, uniform angular arrays (UAA) with an equal angular separation of $2\pi/M$ between two neighbor sensors exist for any number of sensors. In the 3D case, however, there exist only five regular solids offering a perfect symmetry, the so called Platonic solids: tetrahedron, octahedron, cube, icosahedron, and dodecahedron [7]. How to place the sensors in this case if M is different from the number of vertices of the Platonic solids?

Yet another strategy to place the sensors “uniformly” on a D -dimensional unit sphere is to maximize the minimum distance between any pair of sensors. In computational geometry, this is known as the *spherical code* (SC) problem:

$$\max_{\{\underline{g}_i\}} f_{\text{SC}} = \min_{(i,j) \in \mathcal{I}_0} \|\underline{g}_{ij}\|. \quad (7)$$

It is closely related to the spherical packing and kissing number problem [8]. Below we discuss the relationship between these different strategies and the resulting sensor array geometries.

3.1. The two-dimensional case

The first approach in (5) has been studied in [1]. It results in two conditions

$$\underline{g}\underline{1} = \underline{0} \quad \text{and} \quad \underline{g}\underline{g}^T = \frac{M}{D}\mathbf{I} \quad (8)$$

with $\underline{g} = [\underline{g}_1 \dots \underline{g}_M]$ and $\underline{1} = [1, \dots, 1]^T$. \mathbf{I} is an identity matrix. The minimum value of f_{CRB} is $f_{\text{CRB},\min} =$

$(v\sigma)^2(D/M)^2$. It turns out that UAAs and even their centered superpositions satisfy these two conditions and are thus CRB-optimum. But there also exist 2D optimum arrays other than superposition of UAAs. One example is given in Table 2. It contains the direction vectors $\underline{g}_i = [x_i, y_i]^T$ and the corresponding angles α_i .

Sensor i	$x_i = \cos(\alpha_i)$	$y_i = \sin(\alpha_i)$	α_i
1	1	0	0°
2	3/5	$\sqrt{1 - x_2^2}$	$\approx 53^\circ$
3	$(\sqrt{57} - 11)/20$	$\sqrt{1 - x_3^2}$	$\approx 100^\circ$
4	$(-\sqrt{57} - 11)/20$	$\sqrt{1 - x_4^2}$	$\approx 158^\circ$
5	x_4	$-y_4$	$\approx -158^\circ$
6	x_3	$-y_3$	$\approx -100^\circ$
7	x_2	$-y_2$	$\approx -53^\circ$

Table 2. An optimum 7-sensor array in the sense of (5)

Regarding the FIM criterion in (6), we can show

$$\begin{aligned} f_{\text{FIM}} &= \frac{1}{(v\sigma)^2} \text{tr}[\mathbf{G}\mathbf{G}^T] \\ &= \frac{1}{(v\sigma)^2} (M^2 - \|\underline{g}\underline{1}\|^2) \geq \frac{M^2}{(v\sigma)^2} \end{aligned} \quad (9)$$

by using Eq. (14) from [1]. f_{FIM} is maximized if and only if the first condition in (8) is satisfied. Clearly, CRB-optimum sensor arrays are also FIM-optimum, but not vice versa. On the other hand, f_{FIM} is not a meaningful accuracy measure because it does not require the existence of the inverse FIM. A 4-sensor array with $\underline{g}_1 = \underline{g}_2 = -\underline{g}_3 = -\underline{g}_4$, for example, maximizes f_{FIM} , but the FIM \mathbf{J} has rank one and no CRB exists. The reason is the collapse of all direction vectors \underline{g}_i onto one dimension and no information is available for the estimation of the perpendicular components of the source position vector. By using the CRB approach instead of FIM, we explicitly force \mathbf{J} to be a full-rank matrix to prevent \underline{g}_i from collapsing onto a lower-dimensional subspace.

Concerning the spherical code approach in (7), it is obvious that 2D spherical codes are identical to UAAs. Therefore, the relationship between different 2D optimization strategies is

$$\mathcal{A}_{\text{SC}} = \mathcal{A}_{\text{UAA}} \subset \mathcal{A}_{\text{S-UAA}} \subset \mathcal{A}_{\text{CRB}} \subset \mathcal{A}_{\text{FIM}}. \quad (10)$$

The terms \mathcal{A}_{SC} , \mathcal{A}_{UAA} etc. denote the set of all array geometries satisfying the SC, UAA, superposition of UAAs, CRB, and FIM criterion, respectively. \subset is the symbol for subset. As we see, the spherical code strategy always returns UAAs and is more restrictive than the CRB approach.

3.2. The three-dimensional case

In 3D source localization, the CRB and FIM approach in (5) and (6) result in the same optimality conditions in (8) as for the 2D case. In other words, CRB-optimum sensor arrays are

also FIM-optimum and the FIM approach does not guarantee the existence of the CRB. Also Platonic solids as well as their centered superpositions are CRB-optimum [1]. Since there are only five Platonic solids with $M = 4, 6, 8, 12, 20$ vertices (sensors), it is an open question until now how to place the sensors optimally in 3D for odd M .

We guess that no CRB-optimum solutions exist for odd M in the 3D case. Instead, we propose to use spherical codes for the sensor placement. Note that the general problem of spherical codes has not been solved yet for $D \geq 3$ and arbitrary values of M . Nevertheless, Sloane, Hardin, and Smith have found spherical codes and putatively optimum spherical packings for $D = 3, 4, 5$ and $M \leq 130$ [9].

Table 3 compares the Platonic solids and spherical codes. Tetrahedron, octahedron, and icosahedron are both Platonic solids and spherical codes. Differences arise at $M = 8$ and 20. The cube with 6 square faces is a Platonic solid but not a spherical code. The spherical code for $M = 8$ is the square antiprism. It arises from the cube by rotating its top square base by 45° while keeping the bottom square base fixed, thus producing 8 triangular side faces. The distance between the top and bottom base is reduced in such a way that the square antiprism has equal edge length. This spherical code is neither a Platonic solid nor CRB-optimum. Similarly, the dodecahedron with 20 vertices (sensors) is a Platonic solid but not a spherical code. The spherical code for $M = 20$ found in [9] is neither a Platonic solid nor CRB-optimum.

M	Solid	Platonic solids	Spherical codes
4	Tetrahedron	Yes	Yes
6	Octahedron	Yes	Yes
8	Cube Square antiprism	Yes No	No Yes
12	Icosahedron	Yes	Yes
20	Dodecahedron A code from [9]	Yes No	No Yes

Table 3. Platonic solids and spherical codes

Let \mathcal{A}_{PS} and \mathcal{A}_{S-PS} denote the sets of Platonic solids and their centered superpositions. Fig. 1 illustrates the relationship between different 3D sensor placement strategies. It is similar to the 2D relationship in (10) except that \mathcal{A}_{SC} is no longer identical to \mathcal{A}_{PS} . The biggest advantage of spherical codes over Platonic solids is that spherical codes are known for $M \leq 130$. We have compared the localization accuracy f_{CRB} calculated for spherical codes against its minimum value $f_{CRB,\min} = (v\sigma)^2(D/M)^2$:

$$\frac{f_{CRB}(SC)}{f_{CRB,\min}} < 1.04 \quad \text{for } 4 \leq M \leq 30. \quad (11)$$

As we see, though most of the spherical codes are not CRB-optimum in the sense of (5), they are very close to the lower bound $f_{CRB,\min}$. Hence, for any practical number of sensors,

spherical codes can be used to design very good sensor array geometries.

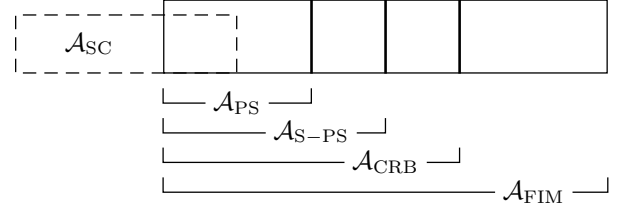


Fig. 1. Relationship between different 3D sensor placement strategies

4. SENSOR PLACEMENT STRATEGIES FOR \mathcal{I}_S

Note that the results derived in the previous section are only valid under the assumption that all sensor pairs are involved in the source localization. If we restrict to the spherical TDOA set \mathcal{I}_s in (4) as requested by many spherical location estimators, the matrix \mathbf{G} in (2) changes. As a consequence, we have to revise our sensor placement strategies.

Intuitively, there is a big difference between the full TDOA set \mathcal{I}_0 and the spherical one \mathcal{I}_s . While all sensors in the former case are equally important, this is not true in the latter case. The reference sensor k in (4) plays a special role because only TDOA measurements relative to it are used in localization. Thus we only expect a symmetry among all sensors excluding the reference one. This means, sensor arrays minimizing f_{CRB} under \mathcal{I}_s will differ from UAAs and Platonic solids. Similarly, the spherical code approach treats all sensors equally and will be sub-optimum for \mathcal{I}_s . Maximizing f_{FIM} in (6) is also not suitable because under \mathcal{I}_s

$$f_{FIM} = \frac{1}{(v\sigma)^2} \text{tr}[\mathbf{G}\mathbf{G}^T] = \frac{1}{(v\sigma)^2} \sum_{i \neq k} \|\underline{g}_i - \underline{g}_k\|^2 \quad (12)$$

will be maximum at $\underline{g}_i = -\underline{g}_k$ for all $i \neq k$. The resulting FIM has rank one. The only meaningful sensor placement strategy is to minimize the trace of the CRB as in (5).

For uncorrelated TDOA errors, the results of extensive numerical minimizations of f_{CRB} under \mathcal{I}_s motivate the *clustering conjecture*: Besides the reference sensor, the remaining $M-1$ sensors build D clusters. All sensors within one cluster share with the same direction vector \underline{g}_i . The result is that each cluster of sensors can be equivalently represented by a single *effective sensor* having the same direction vector but with a reduced variance of TDOA errors. The problem of designing an M -sensor array simplifies to a $(D+1)$ -sensor problem.

Due to limited space, we only discuss the special case $M = LD + 1$ in this paper. Then each of the D clusters consists of L sensors and the effective TDOA error variance per cluster is σ^2/L . The question is how to place the D effective sensors relative to the reference sensor. This problem has been addressed in [2] for the 2D case. The optimum direction vectors for the reference sensor (first column) and the

two effective sensors are

$$\mathbf{g} = \begin{bmatrix} 1 & \cos \alpha & \cos \alpha \\ 0 & \sin \alpha & -\sin \alpha \end{bmatrix}, \quad \alpha = 2 \arcsin \sqrt{\frac{2}{3}} \approx 109.5^\circ. \quad (13)$$

The minimum value of f_{CRB} is

$$2\text{D: } f_{\text{CRB},\min} = (v\sigma)^2 \frac{27}{16(M-1)}. \quad (14)$$

Below we derive the optimum 3D sensor placement. Again the reference sensor is assumed to be located at $\underline{g}_1 = [1, 0, 0]^T$. Since the remaining three effective sensors should be symmetric with respect to both the source and the reference sensor as well as among themselves, they are equally spaced on the thick circle in Fig. 2 which is the intersection of the unit sphere with a cone oriented along the x -axis and having its vertex at \underline{g}_1 . Let \underline{g}_2 be on the xy -plane. The direction vectors \underline{g}_i ($1 \leq i \leq 4$) are then

$$\mathbf{g} = \begin{bmatrix} 1 & \cos \alpha & \cos \alpha & \cos \alpha \\ 0 & \sin \alpha & -\frac{1}{2} \sin \alpha & -\frac{1}{2} \sin \alpha \\ 0 & 0 & \frac{\sqrt{3}}{2} \sin \alpha & -\frac{\sqrt{3}}{2} \sin \alpha \end{bmatrix}. \quad (15)$$

After some straightforward calculations, we obtain

$$f_{\text{CRB}} = (v\sigma)^2 \frac{3t+1}{12t^2(1-t)}, \quad t = \sin^2(\alpha/2). \quad (16)$$

A minimization of f_{CRB} over α returns $\alpha = 2 \arcsin(3^{-1/4}) \approx 98.9^\circ$. The minimum value is

$$3\text{D: } f_{\text{CRB},\min} = (v\sigma)^2 \frac{3(3+2\sqrt{3})}{4(M-1)}. \quad (17)$$

Note that such a sensor array is realizable. Though the sensors within one cluster share with the same direction vector, they can be placed at different distances to the source. The reason for the clustering phenomenon is a fixed reference sensor. Only TDOA measurements relative to it are used in localization. Hence we do not care about $\underline{g}_{ij} = \underline{0}$ within one sensor cluster. This is impossible when we use the full TDOA set \mathcal{I}_0 . Two identical direction vectors with $\underline{g}_{ij} = \underline{0}$ would imply zero contribution to the source localization.

Finally, it is interesting to compare the spherical codes with the CRB-optimum ones. We have calculated the ratio between f_{CRB} for spherical codes and its minimum in (14) and (17):

$$\begin{aligned} 2\text{D: } & \frac{f_{\text{CRB}}(\text{SC})}{f_{\text{CRB},\min}} < 1.6, \quad 3 \leq M = 1 + 2l \leq 130, \\ 3\text{D: } & \frac{f_{\text{CRB}}(\text{SC})}{f_{\text{CRB},\min}} < 1.4, \quad 4 \leq M = 1 + 3l \leq 30. \end{aligned} \quad (18)$$

There is a considerable but bounded performance loss for spherical codes against the CRB-optimum under \mathcal{I}_s .

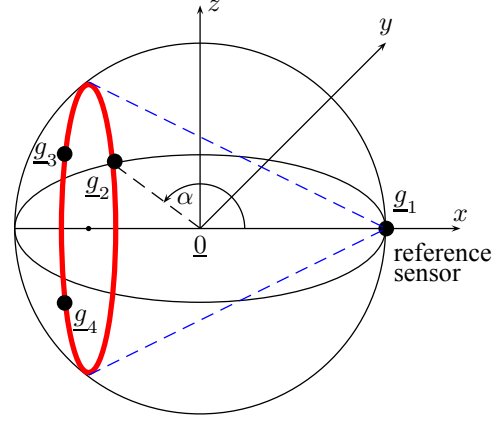


Fig. 2. Optimum geometry of a 4-sensor array under \mathcal{I}_s

5. CONCLUSION

In this paper, we compared different sensor placement strategies for TDOA based localization. We studied the new spherical code approach. It returns nearly CRB-optimum solutions for the open problem “3D, full set \mathcal{I}_0 , and odd number of sensors”. We also derived a new CRB-optimum solution for 3D sensor placement under \mathcal{I}_s . Nevertheless, this study has to be adapted to a more realistic (yet unknown) covariance matrix of the TDOA estimators in the future.

6. REFERENCES

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